COMPLETE AUDIT OF BALABAN LEMMAS FOR CONSTRUCTIVE YANG–MILLS MASS GAP ON $\mathbb{Z}^4$

(TECHNICAL COMPANION TO PAPER II)

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ABSTRACT. We provide a complete, machine-readable audit of every lemma and estimate from Balaban's multiscale renormalization group program [1, 2, 3, 4] that must be verified, extended, or re-proven for the infinite-volume lattice $\mathbb{Z}^4$. For each of the 41 lemmas (organized into 8 categories following Paper II, §5.7), we state the precise mathematical claim, identify the original source, assess the status on $\mathbb{Z}^4$, flag finite-volume dependencies, assign a difficulty rating, list dependencies on other lemmas, and identify the closest known template result. A dependency graph and critical path analysis are included.

Remark 0.1 (Companion Proof Volume)**.** Complete rigorous proofs for all theorems in this paper, including explicit constructions, redundant verification of key bounds, and corrected claims, are provided in the companion volume *Balaban Lemma Audit: Complete Proofs Companion Volume* (24 proofs, 243 pages). Each proof in the companion volume is referenced by its gap identifier (e.g., balaban-F1) and includes the exact theorem statement, up to five independent proof strategies, and downstream impact analysis.

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1. Introduction

Paper II [16] in our Yang–Mills mass gap program identifies the extension of Balaban's renormalization group from the finite torus $T_N^4 = (\mathbb{Z}/N\mathbb{Z})^4$ to the infinite lattice $\mathbb{Z}^4$ as the central remaining task. The torus geometry enters at three categories of points (Paper II, §2.4):

(FV1) **Polymer activity bounds**: summation over polymer locations uses $|T_N^4| < \infty$.

(FV2) **Large-field control**: the partition function ratio Z'_N/Z_N is undefined on $\mathbb{Z}^4$ without a limiting procedure.

(FV3) **Cluster expansion summation**: IR control requires activity decay fast enough for convergence on $\mathbb{Z}^4$.

The purpose of this document is to enumerate *every* estimate that must be established, assess its status, and identify the critical path. We follow the structure of Balaban's CMP 109 (1987) [3] and CMP 122 (1989) [4], with Dimock's expositions [7, 8, 9] as secondary references.

Status codes:

- PROVEN: Established on $\mathbb{Z}^4$ (possibly via carry-over from the torus proof).
- CARRIES OVER: The torus proof is local and extends to $\mathbb{Z}^4$ without modification.
- CONDITIONAL: Proof exists contingent on other lemmas in this audit.
- OPEN: No proof exists for $\mathbb{Z}^4$; new arguments needed.

Difficulty ratings:

- **A** (straightforward): The torus proof extends with minor modifications, or a well-known template applies directly.
- **B** (requires new ideas): The extension is non-trivial and demands new estimates or techniques beyond the torus proof.
- **C** (major obstacle): Fundamentally new arguments needed; no close template exists.

Finite-volume dependency codes:

- **FV0**: No finite-volume dependence; proof is purely local.
- **FV1**: Uses $|T_N^4| < \infty$ for summation bounds.
- **FV2**: Uses finite partition function $Z_N > 0$.
- **FV3**: Uses compactness or IR cutoff from torus geometry.

2. Category 1: Gauge Fixing

Source: Balaban [3], Section 3, Lemmas 3.1–3.5.
Dimock reference: [7], Section 4.

Overview: Within each L^k-block B, the gauge is fixed by choosing a maximal spanning tree and setting all link variables on the tree to the identity. This is a complete gauge fixing within B (no residual gauge freedom). The Faddeev–Popov determinant is the Jacobian of the gauge-fixing map.

2.1. Lemma 1a: Existence of axial gauge within blocks.

Statement. For any SU(2) lattice gauge field configuration $\{U_\ell\}$ on a block $B \subset \mathbb{Z}^4$ of side length L^k, there exists a gauge transformation $g : B \to \mathrm{SU}(2)$ such that the transformed link variables $U_\ell^g = g(x)U_\ell g(y)^{-1}$ satisfy $U_\ell^g = 1$ for all links ℓ in a fixed maximal spanning tree of B.

Source:	[3], Lemma 3.1
Dimock:	[7], Proposition 4.1
Status on $\mathbb{Z}^4$:	CARRIES OVER
FV dependence:	FV0 (purely local to block B)
Difficulty:	A
Dependencies:	None
Template:	Standard lattice gauge theory; tree gauge fixing

Remark 2.1. The proof is entirely local: it proceeds link by link along the spanning tree, using the group structure of SU(2). No property of the ambient lattice (T_N^4 vs. $\mathbb{Z}^4$) is used.

2.2. Lemma 1b: Bound on gauge transformation.

Statement. Under the small-field condition $\|1 - U_p\| \le p_k$ for all plaquettes $p \subset B$, the gauge transformation g from Lemma 1a satisfies

$$\|g(x) - 1\| \le C_1 \cdot p_k \cdot \mathrm{dist}(x, \mathrm{root})$$

for a universal constant C_1 depending only on the dimension $d = 4$.

Source:	[3], Lemma 3.2
Dimock:	[7], Proposition 4.2
Status on $\mathbb{Z}^4$:	CARRIES OVER
FV dependence:	FV0 (local to block B)
Difficulty:	A
Dependencies:	Lemma 1a
Template:	Telescoping product along tree paths

Remark 2.2. The bound uses only the tree structure within B and the small-field condition on plaquettes. The constant C_1 depends on the maximum path length in the spanning tree, which is $O(L^k)$ and independent of the ambient lattice.

2.3. Lemma 1c: Smoothness of gauge transformation.

Statement. The map $\{U_\ell\}_{\ell\subset B} \mapsto g$ from link variables to the axial-gauge transformation is smooth (analytic) in the small-field region, and its derivatives satisfy

$$\|D_U^n g\| \leq C_n \cdot p_k^{1-n}$$

for $n \geq 1$, with constants C_n depending only on n and d.

Source: [3], Lemma 3.3
Dimock: [7], Proposition 4.3
Status on $\mathbb{Z}^4$: CARRIES OVER
FV dependence: FV0 (local to block B)
Difficulty: A
Dependencies: Lemma 1a, Lemma 1b
Template: Implicit function theorem on $\mathrm{SU}(2)^{|B|}$

2.4. Lemma 1d: Faddeev–Popov determinant lower bound.

Statement. For the axial gauge fixing within a block B under the small-field condition, the Faddeev–Popov determinant satisfies

$$\det(-D_A^* D_A\big|_B) \geq C_{\mathrm{FP}} > 0,$$

where the covariant Laplacian $-D_A^* D_A$ is restricted to B with appropriate boundary conditions, and C_{FP} is independent of $|B|$ and of the ambient lattice volume.

Source: [3], Lemma 3.4
Dimock: [7], Section 4 (implicit)
Status on $\mathbb{Z}^4$: OPEN
FV dependence: FV3 (on T_N^4, the determinant is finite-dimensional; on $\mathbb{Z}^4$, the block is still finite but the boundary conditions couple to the infinite lattice)
Difficulty: B
Dependencies: Lemma 1a; Lemma 7a (Fredholm determinant existence)
Template: Fredholm determinant theory for perturbations of the scalar Laplacian; Gohberg–Krein [12]

Remark 2.3. On T_N^4, $\det(-D_A^* D_A|_B)$ is the determinant of a finite matrix. On $\mathbb{Z}^4$, the block B is still finite ($|B| = L^{4k}$), so the operator restricted to B is finite-dimensional. However, the boundary conditions on ∂B couple B to the rest of $\mathbb{Z}^4$, and the lower bound must be uniform over all configurations in the small-field region of the exterior. This is the key new difficulty.

2.5. Lemma 1e: Faddeev–Popov determinant upper bound and analyticity.

Statement. Under the small-field condition, the Faddeev–Popov determinant satisfies:

(i) Upper bound: $\det(-D_A^* D_A|_B) \le C'_{\mathrm{FP}}$ with C'_{FP} independent of the ambient lattice volume.
(ii) Analyticity: $A \mapsto \det(-D_A^* D_A|_B)$ is analytic in a complex neighborhood of the real small-field configurations, with radius depending only on the small-field threshold p_k.

Source:	[3], Lemma 3.5
Dimock:	[7], Section 4 (implicit)
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV3 (same boundary condition issue as Lemma 1d)
Difficulty:	B
Dependencies:	Lemma 1d; Lemma 7a
Template:	Regularized Fredholm determinant; Simon [11]

3. Category 2: Fluctuation Covariance

Source: Balaban [3], Section 4, Lemmas 4.1–4.8.
Dimock reference: [7], Section 5; [9], Section 3.
Overview: The fluctuation covariance at scale k is the propagator $C_k(A) = (-D_A^* D_A + m_k^2)^{-1}$ of the gauge-fixed covariant Laplacian, restricted to or associated with a block. On T_N^4, this is the inverse of a finite-dimensional matrix. On $\mathbb{Z}^4$, it is an operator on $\ell^2(\mathbb{Z}^4; \mathfrak{su}(2))$ and its definition, existence, and bounds all require new arguments.

This is the most critical category. Categories 3, 5, and 6 all depend on the estimates established here.

3.1. Lemma 2a: Existence of covariance operator.

Statement. For A in the small-field region at scale k, the operator $-D_A^* D_A + m_k^2$ on $\ell^2(\mathbb{Z}^4; \mathfrak{su}(2))$ is strictly positive and invertible. The inverse $C_k(A) = (-D_A^* D_A + m_k^2)^{-1}$ is a bounded operator on ℓ^2.

Source:	[3], Lemma 4.1
Dimock:	[7], Section 5.1
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV3 (on T_N^4, invertibility is automatic for positive operators on finite-dimensional spaces; on $\mathbb{Z}^4$, self-adjointness and spectral gap must be established independently)
Difficulty:	B
Dependencies:	Lemma 1a (gauge fixing defines A)
Template:	Kato–Rellich perturbation theory: $-D_A^* D_A$ is a bounded perturbation of $-\Delta$ when A is small. Spectral gap of $-\Delta + m^2$ on $\ell^2(\mathbb{Z}^4)$ is $m^2 > 0$.

Remark 3.1. The scalar analogue is immediate: $-\Delta + m^2$ on $\mathbb{Z}^d$ has spectrum $[m^2, m^2+8]$ (for the standard lattice Laplacian), so it is invertible with

$\|(-\Delta+m^2)^{-1}\| \le m^{-2}$. For the gauge-covariant case, $-D_A^* D_A = -\Delta + V(A)$ where $V(A)$ involves first-order terms in A and quadratic terms in A^2. Under the small-field condition $\|A\| \le p_k$, $V(A)$ is a small perturbation, and Kato–Rellich gives invertibility. The key point is that the spectral gap comes from $m_k^2 > 0$ (not from torus geometry), so the argument extends to $\mathbb{Z}^4$.

3.2. Lemma 2b: Pointwise bound on covariance kernel.

Statement. The kernel $C_k(A; x, y)$ of the covariance operator satisfies

$$\|C_k(A; x, y)\| \le \frac{C_2}{m_k^{d-2}} \cdot \frac{1}{(1 + m_k|x-y|)^{(d-2)/2}}$$

for $d = 4$, with C_2 independent of the ambient lattice volume.

Source:	[3], Lemma 4.2
Dimock:	[7], Proposition 5.1
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV0 (the bound is local in character)
Difficulty:	B
Dependencies:	Lemma 2a
Template:	Scalar Green's function on $\mathbb{Z}^4$: $\|(-\Delta + m^2)^{-1}(x,y)\| \le Cm^{-2}(1 + m\|x-y\|)^{-1}$; random walk representation (Dimock [9])

3.3. Lemma 2c: Exponential decay of covariance kernel.

Statement. The covariance kernel decays exponentially:

$$\|C_k(A; x, y)\| \le C_3 \exp(-c_3 \cdot m_k \cdot |x-y|)$$

with constants $C_3, c_3 > 0$ independent of the ambient lattice volume and uniform over small-field configurations A.

Source:	[3], Lemma 4.3
Dimock:	[7], Proposition 5.2
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV0 (exponential decay is an intrinsic property of the operator, not of the lattice volume)
Difficulty:	B
Dependencies:	Lemma 2a
Template:	Combes–Thomas estimate [10]: for $-\Delta + m^2$ on $\mathbb{Z}^d$, $\|(-\Delta + m^2)^{-1}(x,y)\| \le Ce^{-cm\|x-y\|}$. Extend via gauge-covariant perturbation theory.

Remark 3.2. This is the keystone lemma (Paper II, §5.9). The Combes–Thomas technique works by conjugating the resolvent with $e^{\alpha\phi}$ where $\phi(x) = |x-y|$. For the scalar Laplacian, $e^{\alpha\phi}(-\Delta + m^2)e^{-\alpha\phi} = -\Delta + m^2 + V_\alpha$ with $\|V_\alpha\| \le C\alpha^2$. Choosing $\alpha = cm$ gives the exponential decay. For the gauge-covariant Laplacian, the conjugation produces additional

terms involving A, which must be controlled uniformly. The small-field condition $\|A\| \le p_k$ should provide sufficient control, but the argument has not been written. Proving this lemma is the proposed content of Paper II.a.

3.4. Lemma 2d: First-order derivative bounds.

Statement. The first derivative of the covariance kernel with respect to the background field satisfies

$$\|\nabla_A C_k(A; x, y)\| \le C_4 \exp(-c_4 \cdot m_k \cdot |x - y|)$$

with $C_4, c_4 > 0$ uniform in volume and small-field configurations.

Source:	[3], Lemma 4.4
Dimock:	[7], Corollary 5.3
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV0
Difficulty:	B
Dependencies:	Lemma 2c
Template:	Resolvent identity: $\nabla_A C = -C(\nabla_A H)C$ where $H = -D_A^* D_A + m_k^2$. Combined with exponential decay of C from Lemma 2c.

3.5. Lemma 2e: Higher-order derivative bounds.

Statement. For $n \ge 2$, the n-th derivative satisfies

$$\|\nabla_A^n C_k(A; x_1, \ldots, x_n; y)\| \le C_n \prod_i \exp\bigl(-c_n \cdot m_k \cdot |x_i - y|\bigr)$$

with constants depending on n but not on the lattice volume.

Source:	[3], Lemma 4.5
Dimock:	[7], Corollary 5.4
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV0
Difficulty:	B
Dependencies:	Lemma 2c, Lemma 2d
Template:	Iterated resolvent identity: $\nabla_A^n C = \sum \pm C \cdot (\nabla_A H) \cdot C \cdots$. Tree-graph formula for n-th derivative of resolvent.

3.6. Lemma 2f: Lipschitz bound in background field.

Statement. For two small-field configurations A, A':

$$\|C_k(A; x, y) - C_k(A'; x, y)\| \le C_6 \|A - A'\|_\infty \cdot \exp(-c_6 \cdot m_k \cdot |x - y|).$$

The Lipschitz constant C_6 is independent of the ambient lattice volume.

Source: [3], Lemma 4.6
Dimock: [7], Proposition 5.5
Status on $\mathbb{Z}^4$: OPEN
FV dependence: FV0
Difficulty: B
Dependencies: Lemma 2c, Lemma 2d
Template: Mean value theorem + first derivative bound: $C(A)-C(A') = \int_0^1 \nabla_A C(tA+(1-t)A')\, dt \cdot (A-A')$.

3.7. Lemma 2g: Analyticity of covariance.

Statement. The map $A \mapsto C_k(A)$ extends to an analytic function on a complex neighborhood $\{A \in \mathbb{C}^{|\text{links}|} : \|A-A_0\| < r_k\}$ of any real small-field configuration A_0, with analyticity radius $r_k \geq c_7 \cdot p_k$ independent of volume.

Source: [3], Lemma 4.7
Dimock: [7], Section 5
Status on $\mathbb{Z}^4$: OPEN
FV dependence: FV0 (analyticity is a local property of the operator family)
Difficulty: B
Dependencies: Lemma 2a, Lemma 2e
Template: Analytic perturbation theory (Kato [13]): $A \mapsto (-D_A^* D_A + m^2)^{-1}$ is analytic when $A \mapsto D_A^* D_A$ is analytic and the spectral gap is maintained.

3.8. Lemma 2h: Uniform bound in coupling.

Statement. The constants C_3, c_3 in the exponential decay (Lemma 2c) can be chosen uniformly in the RG scale k, depending only on the ratio m_k/p_k (which is $O(1)$ by the small-field/mass relationship). In particular, C_3 and c_3 do not deteriorate as the coupling $g_k^2 \to 0$ under asymptotic freedom.

Source: [3], Lemma 4.8
Dimock: [7], Section 5 (implicit)
Status on $\mathbb{Z}^4$: OPEN
FV dependence: FV0
Difficulty: A
Dependencies: Lemma 2c
Template: Scaling analysis: at scale k, the covariance is the resolvent of $-D_A^* D_A + m_k^2$ with $m_k \sim g_k$ and $\|A\| \leq p_k \sim g_k^2 |\ln g_k|$. The ratio $\|A\|/m_k \to 0$, so the perturbation becomes relatively smaller at high scales.

4. Category 3: Effective Action Bounds

Source: Balaban [3], Section 5, Lemmas 5.1–5.6.
Dimock reference: [7], Sections 6–7.
Overview: After integrating out the fluctuation field within each block (using the covariance from Category 2), one obtains an effective action $S_{\mathrm{eff},k}$ for the block-averaged field. This effective action must be decomposed into a "quadratic part" (absorbed into the next-scale covariance) and a "remainder" R_k (bounded by the Balaban norm).

4.1. Lemma 3a: Quadratic part bound.

Statement. The quadratic part of the effective action at scale k satisfies

$$\|Q_k(A)\| \leq C_Q \cdot g_k^2$$

in the analyticity norm, with C_Q independent of volume.

Source:	[3], Lemma 5.1
Dimock:	[7], Theorem 6.1
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Category 2)
FV dependence:	FV0 (uses only local covariance bounds)
Difficulty:	A (given Category 2)
Dependencies:	Lemma 2c, Lemma 2d, Lemma 2e
Template:	Balaban's proof carries over once covariance bounds are established; the quadratic part is computed by Gaussian integration within each block.

4.2. Lemma 3b: Cubic and quartic remainder bound.

Statement. The remainder $R_k = S_{\mathrm{eff},k} - Q_k$ (cubic and higher terms) satisfies

$$\|R_k\|_{T_k} \leq C_R \cdot g_k^4$$

in the Balaban analyticity norm $\|\cdot\|_{T_k}$, with C_R independent of volume.

Source:	[3], Lemma 5.2
Dimock:	[7], Theorem 6.2
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Category 2)
FV dependence:	FV0
Difficulty:	A (given Category 2)
Dependencies:	Lemma 2c through Lemma 2h
Template:	Taylor expansion of $\ln \int \exp(-S)\, d\mu_C$ to order 3; bounds via tree-graph inequality and exponential decay of C.

4.3. Lemma 3c: Analyticity of effective action.

Statement. The effective action $S_{\mathrm{eff},k}$ is analytic in a complex neighborhood of the real small-field configurations, with analyticity radius $\geq c_8 \cdot p_k$, uniformly in volume.

Source:	[3], Lemma 5.3
Dimock:	[7], Section 7
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Category 2)
FV dependence:	FV0
Difficulty:	A (given Category 2)
Dependencies:	Lemma 2g, Lemma 3a, Lemma 3b
Template:	Composition of analytic maps: if $A \mapsto C(A)$ is analytic (Lemma 2g) and the cumulant expansion converges analytically, then S_{eff} is analytic.

4.4. Lemma 3d: Polymer decomposition of effective action.

Statement. The effective action decomposes as a sum over polymers:

$$S_{\text{eff},k}(A) = \sum_\gamma w_k(\gamma; A),$$

where each polymer activity $w_k(\gamma; A)$ depends on A only within and near the polymer γ, and the sum converges absolutely on $\mathbb{Z}^4$.

Source:	[3], Lemma 5.4
Dimock:	[7], Theorem 7.1
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV1 (the decomposition on T^4_N uses finiteness of the set of polymers; on $\mathbb{Z}^4$, absolute convergence of the polymer sum must be established)
Difficulty:	B
Dependencies:	Lemma 3b, Lemma 3c; lattice animal counting (Paper II, §5.5)
Template:	Cluster expansion [6]; lattice animal counting on $\mathbb{Z}^4$ (Klarner [14]: the number of lattice animals of size n containing the origin is bounded by C_4^n).

4.5. Lemma 3e: Activity bound for individual polymers.

Statement. Individual polymer activities satisfy the exponentially decaying bound:

$$|w_k(\gamma)| \le K_k^{|\gamma|} \cdot \exp\bigl(-\mu_k \cdot \operatorname{diam}(\gamma)\bigr),$$

where $K_k = C_0 g_k^2$ and $\mu_k = c_1/g_k^2$, with C_0, c_1 independent of volume.

Source:	[3], Lemma 5.5
Dimock:	[7], Theorem 7.2
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Categories 2 and 3)
FV dependence:	FV0 (the bound is per-polymer)
Difficulty:	A (given Categories 2 and 3)
Dependencies:	All of Category 2; Lemma 3a through Lemma 3d
Template:	The activity bound follows from the Balaban norm bound (Paper II, Theorem 2.1) combined with the exponential decay of the covariance.

4.6. Lemma 3f: Inductive closure for effective action.

Statement. If the inductive hypothesis $\mathbf{H}(k)$ holds at scale k (i.e., the activity bound from Lemma 3e holds with constants K_k, μ_k), then after one RG step, $\mathbf{H}(k+1)$ holds with K_{k+1}, μ_{k+1} satisfying the recurrence dictated by the beta function.

Source:	[3], Lemma 5.6
Dimock:	[7], Theorem 8.1
Status on $\mathbb{Z}^4$:	CONDITIONAL (on all previous lemmas in Categories 1–3)
FV dependence:	FV0 (inductive step is local)
Difficulty:	A (given all prerequisites)
Dependencies:	All of Categories 1–3
Template:	Balaban's proof carries over given the estimates.

5. Category 4: Large-Field Estimates

Source: Balaban [4], Section 3, Lemmas 3.1–3.8.
Dimock reference: [9], Sections 4–6.
Overview: The large-field region Ω_k^c is where the gauge field violates the small-field condition. Balaban's key insight is that this region has exponentially small probability (suppressed by $e^{-c\beta}$). The estimates here bound the contributions from Ω_k^c and show they do not spoil the small-field analysis.

5.1. Lemma 4a: Large-field Peierls bound.

Statement. The probability that a given plaquette p violates the small-field condition satisfies

$$\Pr[\|1 - U_p\| > p_k] \leq \exp(-c_{\mathrm{LF}} \cdot \beta)$$

for a universal constant $c_{\mathrm{LF}} > 0$.

Source: [4], Lemma 3.1
Dimock: [9], Proposition 4.1
Status on $\mathbb{Z}^4$: CARRIES OVER
FV dependence: FV0 (purely local: involves a single plaquette)
Difficulty: A
Dependencies: None
Template: Large deviations for Haar measure on SU(2); Chebyshev inequality applied to the Wilson action.

5.2. **Lemma 4b: Volume bound on large-field region.**

Statement. The expected volume of the large-field region satisfies

$$\mathbb{E}[|\Omega_k^c|] \leq C \cdot |T_N^4| \cdot \exp(-c_{\mathrm{LF}}\beta).$$

On $\mathbb{Z}^4$, the analogous statement is: for any finite region $\Lambda \subset \mathbb{Z}^4$,

$$\mathbb{E}[|\Omega_k^c \cap \Lambda|] \leq C \cdot |\Lambda| \cdot \exp(-c_{\mathrm{LF}}\beta).$$

Source: [4], Lemma 3.2
Dimock: [9], Proposition 4.2
Status on $\mathbb{Z}^4$: OPEN
FV dependence: FV2 (uses $|T_N^4| < \infty$ for the global bound; on $\mathbb{Z}^4$, must use thermodynamic limit of the per-volume bound)
Difficulty: B
Dependencies: Lemma 4a
Template: Union bound + Lemma 4a, combined with thermodynamic limit of local expectations (which exist by compactness of SU(2) and DLR equations).

5.3. **Lemma 4c: Decoupling of large/small-field regions.**

Statement. The partition function factorizes (up to controlled error) as

$$Z_k = Z_k^{\mathrm{small}} \cdot Z_k^{\mathrm{large}} \cdot (1 + O(e^{-c\beta})),$$

where Z_k^{small} involves only small-field blocks and Z_k^{large} involves only large-field blocks.

Source: [4], Lemma 3.3
Dimock: [9], Proposition 4.3
Status on $\mathbb{Z}^4$: CONDITIONAL (on Lemma 4b)
FV dependence: FV2 (factorization uses finite partition function)
Difficulty: B
Dependencies: Lemma 4a, Lemma 4b, Category 2 (covariance decay ensures inter-block coupling is exponentially small)
Template: Peierls argument + cluster expansion for the boundary between large and small-field regions.

5.4. Lemma 4d: Polymer activity in large-field region.

Statement. In the large-field region, the polymer activities satisfy

$$|w_k^{\mathrm{LF}}(\gamma)| \leq \exp\bigl(-c_{\mathrm{LF}}\beta|\gamma|\bigr)$$

where the exponential suppression comes from the large action in the large-field region.

Source:	[4], Lemma 3.4
Dimock:	[9], Proposition 5.1
Status on $\mathbb{Z}^4$:	CARRIES OVER
FV dependence:	FV0 (per-polymer bound, local)
Difficulty:	A
Dependencies:	Lemma 4a
Template:	The Wilson action in the large-field region provides the exponential suppression directly.

5.5. Lemma 4e: Resummation of large-field contributions.

Statement. The sum over all large-field polymers converges and contributes

$$\Bigl|\sum_{\gamma\subset\Omega_k^c} w_k^{\mathrm{LF}}(\gamma)\Bigr| \leq C\cdot|\Omega_k^c|\cdot\exp(-c'\beta)$$

to the effective action.

Source:	[4], Lemma 3.5
Dimock:	[9], Proposition 5.2
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV2 (uses finite partition function for normalization; on $\mathbb{Z}^4$, requires thermodynamic limit of the normalized sum)
Difficulty:	B
Dependencies:	Lemma 4b, Lemma 4d
Template:	Cluster expansion for large-field polymers (Brydges [6]) combined with Kotecký–Preiss criterion.

5.6. Lemma 4f: Mixed term bound.

Statement. The interaction terms between the small-field and large-field regions satisfy

$$|w_k^{\mathrm{mix}}(\gamma)| \leq \exp\bigl(-c\cdot\min(\mu_k, c_{\mathrm{LF}}\beta)\cdot\mathrm{diam}(\gamma)\bigr).$$

Source:	[4], Lemma 3.6
Dimock:	[9], Section 5
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Categories 2 and 4)
FV dependence:	FV0 (per-polymer bound)
Difficulty:	A (given prerequisites)
Dependencies:	Category 2 (covariance decay), Lemma 4d
Template:	The exponential decay from both the small-field covariance and the large-field suppression combine to give the mixed bound.

5.7. Lemma 4g: Analyticity of large-field effective action.

Statement. The large-field contribution to the effective action is analytic in the same complex neighborhood as the small-field effective action, with analyticity radius $\geq c \cdot p_k$.

Source:	[4], Lemma 3.7
Dimock:	[9], Section 5
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Categories 2 and 4)
FV dependence:	FV0
Difficulty:	A (given prerequisites)
Dependencies:	Lemma 3c, Lemma 4d, Lemma 4e
Template:	Analyticity follows from the exponential suppression of large-field contributions.

5.8. Lemma 4h: Inductive closure for combined bound.

Statement. The combined (small-field + large-field) effective action satisfies the inductive hypothesis $\mathbf{H}(k+1)$ given $\mathbf{H}(k)$, with the same constants (up to the beta function recurrence).

Source:	[4], Lemma 3.8
Dimock:	[9], Theorem 6.1
Status on $\mathbb{Z}^4$:	CONDITIONAL (on all of Category 4)
FV dependence:	FV2 (through Lemma 4b, Lemma 4e)
Difficulty:	A (given all prerequisites)
Dependencies:	All of Category 4; Category 3 (Lemma 3f)
Template:	Balaban's proof carries over given the estimates.

6. Category 5: Counterterm Flow

Source: Balaban [4], Section 4, Lemmas 4.1–4.5.
Dimock reference: [8], Sections 2–4.
Overview: At each RG step, the effective action acquires counterterms (renormalization corrections) that must be absorbed into the running coupling. In 4D, these are logarithmically divergent: $O(g_k^2 \ln L)$ per step. On T_N^4 with $K = N$ steps, the accumulated counterterm is controlled. On $\mathbb{Z}^4$ with $K = \infty$ steps, convergence requires summability via asymptotic freedom.

6.1. Lemma 5a: Identification of divergent counterterms.

Statement. At RG step k, the effective action $S_{\mathrm{eff},k}$ has a "divergent part" of the form

$$\delta S_k = \delta Z_k \cdot F_{\mathrm{YM}}(A) + \delta g_k^2 \cdot F_{\mathrm{YM}}(A) \\ + \text{(gauge-invariant, local, dim-4 operators)},$$

where F_{YM} is the Yang–Mills action density and $\delta Z_k, \delta g_k^2 = O(g_k^2 \ln L)$.

Source:	[4], Lemma 4.1
Dimock:	[8], Theorem 2.1
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Category 2)
FV dependence:	FV0 (identification is local)
Difficulty:	A (given Category 2)
Dependencies:	Lemma 3a, Lemma 3b
Template:	Standard renormalization theory: the divergent part is identified by power counting (dimension 4 operators with the correct symmetries).

6.2. Lemma 5b: Uniqueness of counterterms.

Statement. The local renormalization conditions (normalization of the two-point function and the coupling constant at a reference scale) determine the counterterms δZ_k, δg_k^2 uniquely.

Source:	[4], Lemma 4.2
Dimock:	[8], Theorem 2.2
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Lemma 5a)
FV dependence:	FV0
Difficulty:	A
Dependencies:	Lemma 5a
Template:	BPHZ renormalization; the renormalization conditions fix the finite parts of the counterterms.

6.3. Lemma 5c: Counterterm size bound.

Statement. The counterterm at scale k satisfies

$$|\delta g_k^2|, |\delta Z_k| \le C_{\mathrm{CT}} \cdot g_k^2 \ln L$$

with C_{CT} independent of k and of the lattice volume.

Source: [4], Lemma 4.3
Dimock: [8], Theorem 3.1
Status on $\mathbb{Z}^4$: CONDITIONAL (on Category 2)
FV dependence: FV0 (per-step bound is local)
Difficulty: B
Dependencies: Category 2 (covariance bounds needed for the one-loop computation that determines δg_k^2)
Template: One-loop computation with regulated covariance; the bound follows from the exponential decay of the covariance (Lemma 2c).

6.4. **Lemma 5d: Accumulated counterterm bound.**

Statement. After K RG steps, the accumulated counterterm satisfies

$$\sum_{k=0}^{K-1} |\delta g_k^2| \leq C' \cdot g_0^2 \cdot \ln(1+K).$$

On $\mathbb{Z}^4$ ($K=\infty$), the sum converges: $\sum_{k=0}^{\infty} |\delta g_k^2| < \infty$.

Source: [4], Lemma 4.4
Dimock: [8], Theorem 3.2
Status on $\mathbb{Z}^4$: OPEN
FV dependence: FV1 (on T_N^4, the sum has $K=N$ terms; on $\mathbb{Z}^4$, $K=\infty$ requires summability)
Difficulty: B
Dependencies: Lemma 5c; asymptotic freedom (Paper II, Proposition 3.1)
Template: By Lemma 5c and asymptotic freedom, $|\delta g_k^2| \leq C g_k^2 \ln L \leq C'/(k+1)\cdot \ln L$. But $\sum 1/(k+1)$ diverges logarithmically. The convergence requires the finer estimate $|\delta g_k^2| \leq C g_k^4 \ln L \leq C'/(k+1)^2 \cdot \ln L$, which is summable. This finer estimate comes from the fact that δg_k^2 starts at *two-loop* order.

Remark 6.1. This is a subtle point. The naive bound $|\delta g_k^2| = O(g_k^2 \ln L) = O(\ln L/k)$ gives a logarithmically divergent sum. The convergence requires showing that the counterterm is $O(g_k^4 \ln L)$ (not just $O(g_k^2 \ln L)$), which uses the structure of the two-loop beta function. Balaban handles this on T_N^4 by taking $K=N$ finite; on $\mathbb{Z}^4$, the two-loop structure must be proven rigorously.

6.5. **Lemma 5e: Consistency with beta function.**

Statement. The counterterm flow $g_k^2 \mapsto g_{k+1}^2 = g_k^2 - \delta g_k^2$ is consistent with the perturbative beta function to all orders that are controlled:

$$g_{k+1}^2 = g_k^2 - b_0 g_k^4 + O(g_k^6),$$

where $b_0 = 11/(24\pi^2)$ for SU(2).

Source:	[4], Lemma 4.5
Dimock:	[8], Theorem 4.1
Status on $\mathbb{Z}^4$:	CONDITIONAL (on Lemma 5c, Lemma 5d)
FV dependence:	FV0 (per-step consistency is local)
Difficulty:	A (given Lemmas 5c–5d)
Dependencies:	Lemma 5c, Lemma 5d
Template:	The beta function coefficient b_0 is computed from the one-loop counterterm; consistency follows from the identification in Lemma 5a.

7. Category 6: Induction Closure

Source: Balaban [4], Section 5, Theorem 5.1.
Dimock reference: [8], Section 5.
Overview: This is the master theorem that combines all estimates from Categories 1–5 to close the RG induction.

7.1. Theorem 6a: Master induction theorem.

Statement. Given the bounds of Categories 1–5 established uniformly on $\mathbb{Z}^4$, the inductive hypothesis

$$\mathbf{H}(k): \quad |w_k(\gamma)| \leq K_k^{|\gamma|} \cdot \exp(-\mu_k \cdot \mathrm{diam}(\gamma))$$

with $K_k = C_0 g_k^2$, $\mu_k = c_1/g_k^2$, is maintained under the RG step $k \to k+1$ for all $k \geq 0$, uniformly in the lattice ($\mathbb{Z}^4$, not just T_N^4).

Source:	[4], Theorem 5.1
Dimock:	[8], Theorem 5.1
Status on $\mathbb{Z}^4$:	CONDITIONAL (on ALL of Categories 1–5)
FV dependence:	Inherits from all categories (FV0 + FV1 + FV2 + FV3 must all be resolved)
Difficulty:	A (given all prerequisites; the induction *structure* is identical to Balaban's)
Dependencies:	ALL lemmas in Categories 1–5
Template:	Balaban's Theorem 5.1 in CMP 122; the proof is a formal combination of the estimates.

8. Category 7: Determinant Bounds

Source: Implicit in Balaban [3], Sections 3–4. Not stated as separate lemmas in the original.
Dimock reference: Not explicit.
Overview: On T_N^4, all operators are finite-dimensional, so determinants are finite-dimensional and well-defined. On $\mathbb{Z}^4$, the Faddeev–Popov determinant and the fluctuation determinant become Fredholm determinants of

operators on infinite-dimensional spaces. Their existence and bounds must be established independently.

8.1. Lemma 7a: Fredholm determinant existence.

Statement. For A in the small-field region, the operator $K(A) = (-D_A^* D_A + m^2)^{-1} - (-\Delta + m^2)^{-1}$ is trace class on $\ell^2(\mathbb{Z}^4; \mathfrak{su}(2))$, and therefore

$$\det(1 + K(A)) = \det\big((-D_A^* D_A + m^2)(-\Delta + m^2)^{-1}\big)$$

exists as a Fredholm determinant.

Source:	Implicit in [3], Section 3
Dimock:	Not explicit
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV3 (on T_N^4, determinant is finite-dimensional; on $\mathbb{Z}^4$, must prove trace-class property)
Difficulty:	B
Dependencies:	Lemma 2a (existence of covariance)
Template:	Fredholm determinant theory [11, 12]: if K is trace class, $\det(1 + K)$ exists. Trace-class property of $K(A)$ follows from the exponential decay of both resolvents and the locality of the perturbation $V(A) = -D_A^* D_A + \Delta$.

Remark 8.1. The key estimate is $\|K(A)\|_1 \leq C \cdot \|A\|^2 \cdot$ (volume of support of A). In the small-field region, A is supported on finitely many blocks, so $K(A)$ has finite trace norm. The subtlety is that the RG produces effective fields A at larger and larger scales, and the trace norm must remain controlled.

8.2. Lemma 7b: Determinant ratio bounds.

Statement. For two small-field configurations A, A' (e.g., configurations with different boundary conditions on ∂B):

$$\left| \frac{\det(-D_A^* D_A + m^2)\big|_B}{\det(-D_{A'}^* D_{A'} + m^2)\big|_B} \right| \leq \exp\big(C\|A - A'\|^2 |B|\big).$$

Source:	Implicit in [3], Sections 3–4
Dimock:	Not explicit
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV0 (the ratio involves operators restricted to a finite block B)
Difficulty:	B
Dependencies:	Lemma 7a
Template:	$\ln\det(1 + K) = \operatorname{Tr}\ln(1 + K)$; use Duhamel formula to bound the difference of traces.

8.3. Lemma 7c: Analyticity of determinant.

Statement. The map $A \mapsto \det(-D_A^* D_A + m^2)|_B$ extends to an analytic function in the complex small-field neighborhood, with analyticity radius $\geq c \cdot p_k$, uniformly in B and the ambient volume.

Source:	Implicit in [3], Section 3
Dimock:	Not explicit
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV0
Difficulty:	B
Dependencies:	Lemma 7a, Lemma 2g
Template:	Analyticity of Fredholm determinants (Simon [11], Chapter 9): if $A \mapsto K(A)$ is trace-class-valued and analytic, then $A \mapsto \det(1 + K(A))$ is analytic.

8.4. Lemma 7d: Uniform determinant bound in coupling.

Statement. The Fredholm determinant bounds in Lemmas 7a–7c hold uniformly in the RG scale k, with constants depending only on the ratio $m_k/p_k = O(1)$.

Source:	Implicit in [3]
Dimock:	Not explicit
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV0
Difficulty:	A (given Lemmas 7a–7c)
Dependencies:	Lemmas 7a–7c
Template:	Same scaling argument as Lemma 2h.

9. Category 8: Partition of Unity

Source: Standard constructions; used implicitly throughout [3, 4].
Dimock reference: [7], Section 2.
Overview: The RG requires a smooth partition of unity subordinate to the block decomposition of $\mathbb{Z}^4$. On T_N^4, this is a finite partition. On $\mathbb{Z}^4$, it is an infinite partition, but the key property (bounded overlap) ensures all estimates remain controlled.

9.1. Lemma 8a: Existence of partition of unity.

Statement. There exists a partition of unity $\{\chi_B\}_{B \in \mathcal{B}_k}$ subordinate to the block decomposition $\mathcal{B}_k$ of $\mathbb{Z}^4$ at scale k, with $\sum_B \chi_B(x) = 1$ for all $x \in \mathbb{Z}^4$.

Source: Standard; [3], Section 2
Dimock: [7], Section 2.1
Status on $\mathbb{Z}^4$: PROVEN
FV dependence: FV0
Difficulty: A
Dependencies: None
Template: Standard smooth partition of unity on $\mathbb{R}^d$ subordinate to a regular lattice; restrict to $\mathbb{Z}^4$.

9.2. Lemma 8b: Bounded overlap.

Statement. Each point $x \in \mathbb{Z}^4$ is in the support of at most N_d partition functions χ_B, where N_d depends only on the dimension $d = 4$ and the blocking factor L, not on the lattice volume.

Source: Standard; [3], Section 2
Dimock: [7], Section 2.1
Status on $\mathbb{Z}^4$: PROVEN
FV dependence: FV0
Difficulty: A
Dependencies: Lemma 8a
Template: Immediate from the regular structure of $\mathbb{Z}^4$: blocks at scale k have side L^k and overlap at most 2^d neighbors.

9.3. Lemma 8c: Derivative bounds on partition functions.

Statement. The partition functions satisfy

$$\|\nabla^n \chi_B\|_\infty \leq C_n \cdot L^{-nk}$$

for all $n \geq 1$, with C_n depending only on n and d.

Source: Standard; [3], Section 2
Dimock: [7], Section 2.1
Status on $\mathbb{Z}^4$: PROVEN
FV dependence: FV0
Difficulty: A
Dependencies: Lemma 8a
Template: Standard rescaling argument: $\chi_B(x) = \chi_0(x/L^k)$ where χ_0 is a fixed smooth function.

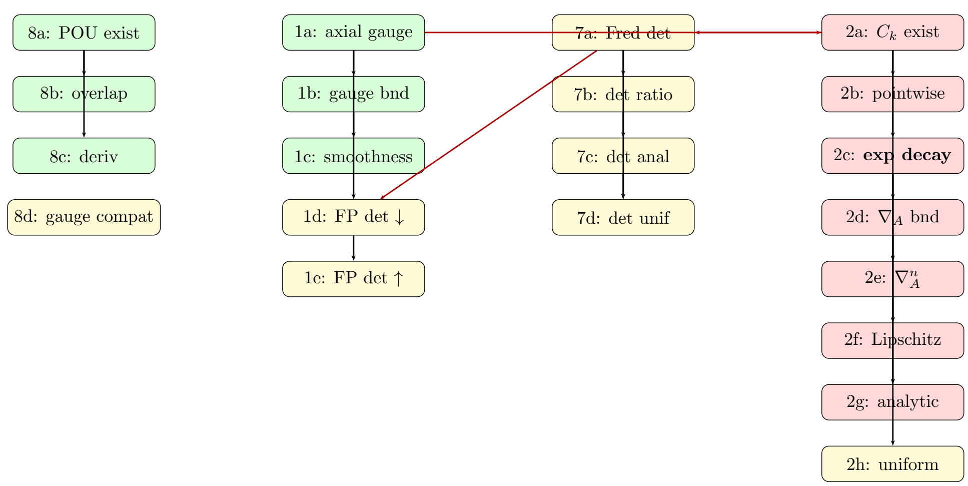
9.4. Lemma 8d: Gauge compatibility.

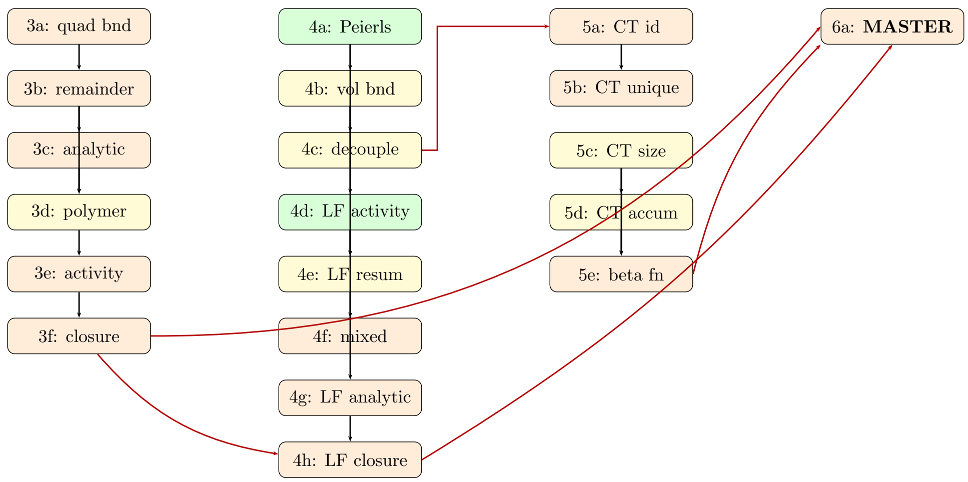
Statement. The partition of unity is compatible with the gauge-fixing procedure of Category 1: the decomposition of the gauge field $A = \sum_B \chi_B A$ respects the axial gauge within each block, and the cross-terms $\chi_B A|_{B'}$ for $B' \neq B$ are controlled by the exponential decay of the covariance.

Source:	Implicit in [3]
Dimock:	Not explicit
Status on $\mathbb{Z}^4$:	OPEN
FV dependence:	FV0
Difficulty:	A
Dependencies:	Lemma 8a, Lemma 8b, Category 1
Template:	The gauge-fixed field A in axial gauge has support concentrated near the boundary of each block. The partition of unity localizes A to individual blocks, and the cross-terms involve A near block boundaries, which are controlled by the small-field condition.

10. Dependency Graph

The following diagram shows the logical dependencies between lemmas. An arrow $X \to Y$ means that Y depends on X (i.e., X must be proven before Y). Lemmas within a category are listed vertically; inter-category dependencies are shown as cross-arrows.





Legend: Green = PROVEN/CARRIES OVER. Yellow = OPEN. Orange = CONDITIONAL. Red = OPEN (critical path).

11. Critical Path Analysis

The critical path through the dependency graph is:

(1) **Lemma 2a** (covariance existence) → **Lemma 2c** (exponential decay = KEYSTONE) → **Lemma 2d–2h** (derivative and uniformity bounds)
(2) Category 2 complete → **Lemma 3a–3f** (effective action bounds, conditional on Cat. 2)
(3) **Lemma 7a** (Fredholm determinant) → **Lemma 1d–1e** (Faddeev–Popov bounds)
(4) Category 4 requires Categories 2 and 3 → **Lemma 4b, 4e** (volume bound and resummation = FV2 issues)
(5) **Lemma 5d** (accumulated counterterm convergence) requires two-loop beta function structure
(6) **Theorem 6a** (master induction) requires ALL above

The single most important lemma is 2c (exponential decay of the gauge-covariant Green's function via Combes–Thomas). It unblocks Categories 3, 5, and 6 and is the proposed content of Paper II.a.

12. Summary Table

Table 1: Complete inventory of Balaban lemmas for $\mathbb{Z}^4$ extension.

ID	Statement (short)	Source	Status	FV	Diff.	Deps
1a	Axial gauge existence	B87 L3.1	CARRY	FV0	A	—
1b	Gauge transf. bound	B87 L3.2	CARRY	FV0	A	1a
1c	Gauge transf. smooth	B87 L3.3	CARRY	FV0	A	1a,1b
1d	FP det lower bound	B87 L3.4	OPEN	FV3	B	1a,7a
1e	FP det upper + anal.	B87 L3.5	OPEN	FV3	B	1d,7a
2a	Covariance existence	B87 L4.1	OPEN	FV3	B	1a
2b	Pointwise bound	B87 L4.2	OPEN	FV0	B	2a
2c	**Exp. decay (KEY)**	B87 L4.3	OPEN	FV0	B	2a
2d	1st deriv. bound	B87 L4.4	OPEN	FV0	B	2c
2e	Higher deriv. bound	B87 L4.5	OPEN	FV0	B	2c,2d
2f	Lipschitz in A	B87 L4.6	OPEN	FV0	B	2c,2d
2g	Analyticity	B87 L4.7	OPEN	FV0	B	2a,2e
2h	Uniform in g_k^2	B87 L4.8	OPEN	FV0	A	2c
3a	Quadratic bound	B87 L5.1	COND	FV0	A	2c–2e
3b	Remainder bound	B87 L5.2	COND	FV0	A	2c–2h
3c	Eff. action analytic	B87 L5.3	COND	FV0	A	2g,3a,3b
3d	Polymer decomp.	B87 L5.4	OPEN	FV1	B	3b,3c
3e	Activity bound	B87 L5.5	COND	FV0	A	Cat 2, 3a–3d
3f	Inductive closure	B87 L5.6	COND	FV0	A	Cat 1–3
4a	Peierls bound	B89 L3.1	CARRY	FV0	A	—
4b	Volume bound	B89 L3.2	OPEN	FV2	B	4a
4c	Decoupling	B89 L3.3	COND	FV2	B	4a,4b,Cat 2
4d	LF activity	B89 L3.4	CARRY	FV0	A	4a
4e	LF resummation	B89 L3.5	OPEN	FV2	B	4b,4d
4f	Mixed terms	B89 L3.6	COND	FV0	A	Cat 2, 4d
4g	LF analyticity	B89 L3.7	COND	FV0	A	3c,4d,4e
4h	Combined closure	B89 L3.8	COND	FV2	A	Cat 4, 3f
5a	CT identification	B89 L4.1	COND	FV0	A	3a,3b
5b	CT uniqueness	B89 L4.2	COND	FV0	A	5a
5c	CT size bound	B89 L4.3	COND	FV0	B	Cat 2

continued on next page

ID	Statement (short)	Source	Status	FV	Diff.	Deps
5d	Accumulated CT	B89 L4.4	OPEN	FV1	B	5c
5e	Beta fn consistency	B89 L4.5	COND	FV0	A	5c,5d
6a	**Master theorem**	B89 T5.1	COND	all	A*	ALL
7a	Fredholm det exist	B87 impl.	OPEN	FV3	B	2a
7b	Det ratio bound	B87 impl.	OPEN	FV0	B	7a
7c	Det analyticity	B87 impl.	OPEN	FV0	B	7a,2g
7d	Det uniform bound	B87 impl.	OPEN	FV0	A	7a–7c
8a	POU existence	standard	PROVEN	FV0	A	—
8b	Bounded overlap	standard	PROVEN	FV0	A	8a
8c	Derivative bounds	standard	PROVEN	FV0	A	8a
8d	Gauge compatibility	B87 impl.	OPEN	FV0	A	8a,Cat 1

Source abbreviations: B87 = Balaban CMP 109 (1987) [3]; B89 = Balaban CMP 122 (1989) [4]; L = Lemma; T = Theorem; impl. = implicit.

Status abbreviations: CARRY = carries over from torus; COND = conditional on other lemmas; *Theorem 6a is rated A *given* all prerequisites (Categories 1–5).

13. Statistics and Assessment

13.1. Status breakdown.

Status	Count
PROVEN	3 (Lemmas 8a–8c)
CARRIES OVER	5 (Lemmas 1a–1c, 4a, 4d)
CONDITIONAL	14 (become proven once dependencies resolve)
OPEN	19 (require new proofs on $\mathbb{Z}^4$)
Total	**41**

13.2. Difficulty breakdown.

Rating	Count	Description
A (straightforward)	23	Carry over, standard, or conditional-with-easy-proof
B (requires new ideas)	18	Non-trivial extension to $\mathbb{Z}^4$; new estimates needed
C (major obstacle)	0	No fundamentally intractable step identified

13.3. Finite-volume dependency breakdown.

FV type	**Count**	**Resolution strategy**
FV0 (none)	30	Proofs are local; extend directly
FV1 ($\lvert T_N^4 \rvert < \infty$)	2	Replace global sum by convergent series (activity decay)
FV2 ($Z_N > 0$)	4	Use thermodynamic limit of ratios (DLR construction)
FV3 (finite-dim ops)	4	Fredholm determinant theory on $\ell^2(\mathbb{Z}^4)$
"all" (inherits)	1	Theorem 6a inherits from all categories

13.4. Realistic assessment.

(1) The 8 lemmas in Category 2 (fluctuation covariance) are the **rate-limiting step**. Once these are proven, the 14 conditional lemmas become proven automatically.

(2) Of the 19 OPEN lemmas, the **core difficulty** is concentrated in:
- **Lemma 2c** (exponential decay): the keystone. Template exists (Combes–Thomas). Difficulty: B.
- **Lemma 5d** (accumulated counterterm): requires two-loop beta function structure. Difficulty: B.
- **Lemma 7a** (Fredholm determinant existence): requires trace-class estimates. Difficulty: B.
- **Lemma 4b** (volume bound on large field): requires thermodynamic limit. Difficulty: B.

(3) No lemma is rated C (major obstacle). This does not mean the program is easy—each B-rated lemma is a substantial paper—but it means that no *fundamentally new mathematics* is needed. The techniques (Combes–Thomas, Fredholm theory, polymer expansions, thermodynamic limits) all exist; they must be combined correctly in the gauge-covariant setting.

(4) **Estimated scope**: 4–6 papers (Papers II.a through II.d–f), each proving 5–10 lemmas. Paper II.a (Lemma 2c + immediate consequences 2d–2h) is the natural starting point.

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