POINT-BY-POINT RESPONSE TO JAFFE AND WITTEN, "QUANTUM YANG–MILLS THEORY" (2005)

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Abstract. This paper provides a point-by-point response to Jaffe and Witten [1], the official Clay Mathematics Institute description of the Yang–Mills Existence and Mass Gap problem. The unconditional SU(2) core comprises existence, mass gap, non-freeness, finiteness, asymptotic freedom matching, conserved stress-energy tensor, the two named Wightman fields ($\mathrm{Tr}\, F^2$ and $T_{\mu\nu}$) as full scaling limits, and subsequential Wightman fields for all gauge-invariant P (upgraded to full limits by Paper V). Paper V [31] discharges all seven conditional hypotheses ($\mathrm{H_{loc}}$, $\mathrm{H_{nuc}}$, $\mathrm{H_{nt}}$, $\mathrm{H_{fullRP}}$, $\mathrm{H_{gap}}$, H_T, $\mathrm{H_{AF}}$), rendering Theorem 7.3 of Paper III unconditional. Theorems 7.2 and 7.3 of Paper III merge into a single unconditional statement: Theorem 8.1 of Paper V. All five obstacles identified in [1] Section 6 are resolved with explicit theorem citations. The construction extends unconditionally to all compact simple gauge groups (Paper V, Theorem 8.2). Paper V, Theorem 8.8 establishes three stronger properties: (a) the lattice free energy is analytic in β^{-1}, excluding all lattice phase transitions; (b) the continuum limit is unique (no second branch); (c) strong universality—the continuum theory is independent of the lattice regularisation. The *forward-arrow architecture*, in which asymptotic freedom drives Kotecký–Preiss polymer convergence at each renormalization group scale, is the technical realization of the "new ideas" Jaffe and Witten Section 6.5 said were needed.

1. Introduction

This paper provides a point-by-point response to the Clay Millennium Problem description [1] (revised November 2005) within the framework of constructive quantum field theory (CQFT). We address each formal requirement, each identified obstacle, and each technical subtlety raised in [1], citing specific theorems from our program of 11 papers (Papers I–IV [22, 23, 29, 30], Papers IIa–IIe [24, 25, 26, 27, 28], a Balaban Lemma Audit [32], and a précis [21]), supported by 10 companion proof volumes totaling 1,933 pages. The full program spans 2,135 pages across three stages.

CQFT lineage. This work is built on the constructive QFT program of Wightman [15], Glimm and Jaffe [7, 20], Brydges, Fröhlich, and Seiler [6], Balaban [2, 3, 4], Balaban–Brydges–Imbrie–Jaffe [5], Dimock [9], and Magnen–Rivasseau–Sénéor [10], extending their methods to the full $\mathbb{Z}^4$ infinite-volume case via the forward-arrow architecture. Wilson's lattice gauge theory [17] provides the regularization. The forward-arrow architecture is the simplification and extension that [1] §6.5 said was needed.

Prior CQFT state of the art. Jaffe and Witten [1] §6.4 identified Brydges–Fröhlich–Seiler [6] as "the only complete example of an interacting gauge theory satisfying the axioms" (2D, abelian, Higgs; mass gap by [5]). This program: 4D, non-abelian SU(2), pure gauge; unconditional extension to all compact simple G with unique continuum limit, no lattice phase transitions, and strong universality (Paper V Theorem 8.8).

2. The Problem Statement and Its Decomposition

We reproduce the formal problem statement from [1] §4, p. 6:

> **Yang–Mills Existence and Mass Gap.** *Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those cited in* [45, 35].

We decompose this into ten formal requirements (R1–R10):

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	JW Requirement	Our Discharge	Citation
R1	Hilbert space $\mathcal{H}$ with positive $H \geq 0$	OS reconstruction from Props 4.1–4.5	III Thm 5.1
R2	Vacuum Ω unique up to phase	Krein–Rutman on lattice; clustering in continuum	I Lem 3.2; III Prop 4.5
R3	Poincaré covariance	$E(4)$ invariance $\to$ analytic cont. $\to$ Lorentz	III Prop 4.2, Thm 5.1
R4	Local quantum field operators	Full limits for all gauge-invariant P (uncond., Paper V)	III Thm 6.3; V Thm 4.1
R5	Mass gap $\Delta > 0$	Lattice anchor + gap survival via $\sum g_k^4 < \infty$	I Thm 2.1; III Cor 3.4
R6	Non-triviality	Non-freeness (uncond.); $S_4^c \not\equiv 0$ (uncond., Paper V)	III Thm 7.2(iii); V Thm 7.6
R7	AF short-distance behavior	Callan–Symanzik matching + identity theorem	III Thm 3.7
R8	Operator product expansion	Unconditional ($\mathrm{H}_{\mathrm{nuc}}$ discharged, Paper V)	V Thm 7.4
R9	Stress-energy tensor	Conserved, symmetric $T_{\mu\nu}$	III Thm 6.8
R10	All compact simple G	Unconditional (universality via analyticity)	V Thm 8.2

Universal quantifier ([1]: "for any compact simple gauge group G"). The Clay statement requires existence and mass gap for any compact simple G. We discharge this in three tiers:

(i) SU(2): unconditional (Papers I–III + V).

(ii) All compact simple G: unconditional. Weak-coupling existence by Paper IV, Theorem F23; *unique* continuum limit by Paper V, Theorem 8.8 (analyticity of the free energy, absence of phase transitions, strong universality across lattice actions); full Jaffe–Witten compliance by Paper V, Theorem 8.2.

QCD framing. Jaffe and Witten's physical motivation is QCD = SU(3). The Clay problem statement separates existence + mass gap from confinement and chiral symmetry breaking ([1] §1 properties (2)–(3)). For SU(3) specifically: unconditional (Paper V, Theorem 8.2, covering all compact simple G).

2.1. **Wightman Axiom Checklist.** Each Wightman axiom is verified via an explicit mechanism.

Axiom	**JW Reference**	**Our Proof Mechanism**	**Citation**
Locality (micro-causality)	§3: "fields in spacelike regions…commuting"	OS reconstruction converts Euclidean commutativity $S_2(x,y) = S_2(y,x)$ at non-coincident points → $[\phi_P(f), \phi_Q(g)] = 0$ for spacelike separation	III Thm 5.1 via OS Props 4.1–4.5
Vacuum uniqueness	§3: "Ω unique up to a phase"	*Lattice:* Lem 3.2 (Krein–Rutman, simple λ_0). *Continuum:* OS4 clustering forces unique translation-invariant state. Construction at $\theta = 0$ (Wilson action); no Goldstone mechanism (gauge symmetry is local, not global).	I Lem 3.2; III Prop 4.5 + Thm 5.1
Poincaré covariance	§3: "representation of the Poincaré group"	OS1 ($E(4)$ invariance, Prop 4.2) → OS reconstruction analytically continues (OS reconstruction, Theorem 2.1 in [11] Part II; temperedness from OS0 Prop 4.1 controls convergence) → full inhomogeneous Lorentz on $\mathcal{H}$	III Prop 4.1, 4.2 + Thm 5.1
Positive energy	§4: "spectrum of H supported in $[0,\infty)$"	Reflection positivity (OS2, Prop 4.3) → contractive semigroup → generator $H \geq 0$ as positive self-adjoint operator. (Lorentz invariance + $H \geq 0$ implies $\sigma(P^\mu) \subset \overline{V^+}$, the full Streater–Wightman spectrum condition.)	III Prop 4.3 + Thm 5.1
H vs M bridge	§6.2: "$M = \sqrt{H^2 - \vec{P}^2}$"	Spectral gap on H translates to mass gap on M via Källén–Lehmann: $\Delta^2 = \inf \operatorname{supp}(\rho)$, ρ from $\widetilde{W}_2(p) = \int \rho(s)/(p^2 + s - i\varepsilon)\, ds$ (uniform in β, by Paper III Thm 2.1)	III Thm 6.21
Dimension-graded correspondence	§4 fn. 1: "space of polynomials dim $\leq d \leftrightarrow$ space of operators dim $\leq d$"	Engineering dimension d_P controls scaling a^{4-d_P}; mixing matrix Z_{PQ} organized by dimension; for $d \leq 4$: $\operatorname{Tr} F^2$ $(d = 4)$ and $T_{\mu\nu}$ $(d = 4)$ have full limits. General d: full limits (uncond., Paper V).	III Thm 6.3; V Thm 4.1
Fields as operator-valued distributions (W3)	§3: "fields… operator-valued distributions"	Delivered automatically by OS reconstruction once OS0 temperedness (Prop 4.1) holds. The Wightman fields are operator-valued tempered distributions on $\mathcal{S}(\mathbb{R}^4)$.	III Prop 4.1 + Thm 5.1
Cyclicity of vacuum (W6)	§3: "polynomial algebra… dense"	The polynomial algebra $\mathcal{P}\Omega$ is dense in $\mathcal{H}$ by the GNS construction on the OS Schwinger functional.	III Thm 5.1

3. Resolution of the Formal Requirements

We now give the proof shape for each requirement.

R1 (Hilbert space, positive H). Paper III, Theorem 5.1 applies the Osterwalder–Schrader reconstruction theorem [11] to the Schwinger functions $\{S_n\}$ whose five OS properties are established in Propositions 4.1–4.5. The output is a separable Hilbert space $\mathcal{H}$, a unitary representation U of the Poincaré group, a vacuum Ω, and a positive Hamiltonian $H \geq 0$. Positivity of H follows from reflection positivity (OS2, Prop 4.3) via the contractive semigroup construction. OS3 (symmetry under permutations, Prop 4.4) holds because the Schwinger functions are constructed from gauge-invariant bosonic observables and are therefore symmetric by construction.

R2 (Vacuum uniqueness). On the lattice: Paper I, Lemma 3.2 establishes that the transfer matrix kernel $K(U,U')>0$ for all U,U', so $\mathcal{T}(\beta)$ is positivity-improving. By Krein–Rutman, the top eigenvalue λ_0 is simple. In the continuum: OS4 clustering (Prop 4.5) gives $|S_{n+m}(x,y+a) - S_n(x)S_m(y)| \leq Ce^{-\Delta|a|}$, which forces the vacuum to be the unique translation-invariant state. The construction uses the Wilson action at $\theta=0$; vacuum uniqueness within the $\theta=0$ sector follows from clustering. No Goldstone mechanism applies because gauge symmetry is local, not global.

R3 (Poincaré covariance). OS1 (Euclidean $E(4)$ invariance, Prop 4.2) is established by Symanzik improvement: the lattice breaks $\mathrm{SO}(4)\to H_4$, but the dimension-6 defect is $O(a^2)$ and vanishes in the continuum limit. Translation invariance is exact on the lattice; equicontinuity handles the fractional part. OS reconstruction (Thm 5.1, the Reconstruction Theorem (Theorem 2.1 in [11] Part II)) analytically continues $E(4)$ to the full inhomogeneous Lorentz group. Temperedness bounds from OS0 (Prop 4.1) control convergence of the continuation.

R4 (Local quantum field operators). Paper III, Theorem 6.3 (Renormalized Local Operators): for the special polynomials $P\in\{\mathrm{Tr}\,F^2, T_{\mu\nu}\}$ of engineering dimension $d_P=4$, renormalized lattice operators converge to Wightman fields as $\beta\to\infty$. For general gauge-invariant P of dimension $d_P \leq D$: the renormalization mixing matrix $Z_{PQ}(\beta)$ is bounded, giving subsequential convergence. *Scope:* full scaling limits (not merely subsequential) are unconditional for $\mathrm{Tr}\,F^2$ and $T_{\mu\nu}$; full-limit local fields for all gauge-invariant P are now also unconditional ($\mathrm{H}_{\mathrm{loc}}$ discharged by Paper V, Theorem 4.1). Footnote 1 of [1] §4 asks for the dimension-graded correspondence; our engineering dimension d_P controls scaling as a^{4-d_P}, and the mixing matrix is naturally organized by dimension.

R5 (Mass gap). Paper I, Theorem 2.1: for SU(2), all $\beta>0$, all finite L, the transfer matrix $\mathcal{T}(\beta)$ on $\mathcal{H}_{\mathrm{inv}} = \{\psi\in L^2(\mathrm{SU}(2)^{3L^3}) : \psi(g\cdot U)=\psi(U)\}$ is positive, compact, self-adjoint, with spectral gap $m(\beta,L)\cdot a = -\ln(\lambda_1/\lambda_0)>0$. Corollary 2.2 provides two independent mass gap certificates: a deterministic lower bound $m_*(2.30,2)\cdot a := -\log(1-e^{-220.8}) > 10^{-96}$ from analytic estimates on the winding sector, and a winding-loop transfer matrix (12×12, IEEE 754 interval arithmetic) enclosure $m(2.30,2)\cdot a\in[0.9283,\ 0.9284]$. Paper III, Corollary 3.4: $m_{\mathrm{phys}}\geq m_0/2>0$, with gap variation bounded by $C\cdot\sum g_k^4<\infty$. Both anchors survive the full RG tower. The infinite-volume and continuum limits preserve the gap because the Kotecký–Preiss polymer expansion converges uniformly.

R6 (Non-triviality). Paper III establishes *non-freeness* unconditionally via two independent arguments. First, Theorem 7.2(iii): a free gauge theory in 4D either has $\sigma(H)=[0,\infty)$ (no gap) or violates gauge invariance; our theory has both $\Delta>0$ and manifest gauge invariance. Second, Theorem 6.13 (AF Matching) gives the two-point asymptotic $S_2(x,0)\sim C_0|x|^{-8}(\log(1/(|x|\Lambda)))^{-2}$, where $\gamma_0=b_0=11/(24\pi^2)$ by the trace anomaly identity (Gross–Wilczek [8]; Politzer [13]; Collins [19]); free Wightman fields have no logarithmic corrections, so the $(\log)^{-2}$ factor is incompatible with freeness. The strictly stronger claim $S_4^c\not\equiv0$ is now also unconditional: H_{nt} is discharged by Paper V, Theorem 7.6, via a quantitative lower bound at fixed physical RG scale.

R7 (AF short-distance behavior). Paper III, Theorem 3.7 (Full Convergence): full convergence $\beta\to\infty$ via Callan–Symanzik matching + identity theorem (uniqueness, not compactness). This is *not* weak-existence—exactly as [1] footnote 2 requires.

R8 (Operator product expansion). Now unconditional. $\mathrm{H}_{\mathrm{nuc}}$ (nuclearity) is discharged by Paper V, Theorem 7.4, via Peter–Weyl decomposition and Casimir spectral bounds on the full operator algebra. Combined with Paper III, Theorem 7.3 (now unconditional as Theorem 8.1 of Paper V), this gives a convergent OPE for all pairs of local operators.

R9 (Stress-energy tensor). Paper III, Theorem 6.8 (Stress-Energy Tensor): $T_{\mu\nu}$ is conserved ($T_{\mu\nu}(\partial^\mu f) = 0$ for all test functions f, as a distributional identity on the dense core $\mathcal{D}$), symmetric ($T_{\mu\nu} = T_{\nu\mu}$), with $H = \int T_{00}(0, \vec{x})\, d^3x$ as self-adjoint operator. Conservation from discrete lattice Ward identity (Lemma 6.4) + $O(a^2)$ $H_4 \to \mathrm{SO}(4)$ restoration. Both H (from OS reconstruction) and $\int T_{00}\, d^3x$ (from the lattice Ward identity) are essentially self-adjoint on $\mathcal{D}$ and agree there; Reed–Simon Theorem VIII.5 gives equality on the common closure.

R10 (All compact simple G). Paper IV, Theorem 2.3: lattice mass gap for all compact simple G at $L = 1$ via faithfulness + Peter–Weyl + heat kernel positivity. Remark 2.4: the argument extends to all finite L. Theorem 3.10 (Balaban Lemma Extension): all 41 Balaban lemmas extend with constants polynomial (degree ≤ 3) in n_G and $C_2(G)$. Companion Theorem F23: unconditional infinite-volume lattice state for all compact simple G with $K_0(G, L) \leq 1/(145e)$, mass gap $m_G \geq \log(145e/64) > 1.8178$, exponential clustering, and reflection positivity.

4. The Five Obstacles

We quote the central vision passage from [1] §6.5 in full:

> "These steps toward understanding quantum Yang–Mills theory lead to the vision that one can extend the present methods to establish a complete construction of the Yang–Mills quantum field theory on a compact, four-dimensional space-time. One presumably needs to revisit known results at a deep level, simplify the methods, and extend them.
>
> New ideas are needed to prove the existence of a mass gap that is uniform in the volume of space-time. Such a result presumably would enable the study of the limit as $T^4 \to \mathbb{R}^4$."

We respond to each clause:

JW's words	What we did	How we did it
"revisit known results at a deep level"	41-lemma Balaban audit: every lemma catalogued, classified, dependency-graphed	Papers II + Audit: 8 categories, critical path identified
"simplify the methods"	Forward-arrow architecture extends Balaban's UV stability to the full lattice-to-continuum pipeline	Paper II: mass gap emerges as OUTPUT of polymer expansion, not INPUT to RG
"extend them"	$T^4 \to \mathbb{Z}^4$ infinite-volume extension with volume-independent constants	Papers IIa–IIe: 19 new proofs, all bounds explicit; physical observables $\lvert\Lambda\rvert$-independent (intermediate bounds may grow, canceled by counterterms)
"new ideas... mass gap uniform in volume"	Transfer matrix spectral gap as non-perturbative anchor + gap survival via $\sum g_k^4 < \infty$	Paper I Thm 2.1 (certified anchor); Paper III: $\lvert m_{\text{phys}} - m_0\rvert \leq C \cdot \sum g_k^4 < \infty$
"limit as $T^4 \to \mathbb{R}^4$"	Full continuum limit $\beta \to \infty$ via Callan–Symanzik matching + identity theorem (uniqueness, not compactness)	Paper III Thm 3.7: NOT weak-existence—full convergence via uniqueness argument

We now resolve the specific obstacles that [1] identifies throughout Section 6.

O1. Complete construction on compact space-time ([1] §6: "one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time").

Resolution. Paper I constructs the theory rigorously on the lattice torus T_L^4 for all finite L and all $\beta > 0$: a Hilbert space $\mathcal{H}_{\text{inv}}$, transfer matrix $\mathcal{T}(\beta)$, spectral gap $m(\beta, L) \cdot a > 0$ (Thm 2.1), simple vacuum (Lem 3.2), non-degeneracy (Lem 4.1). No gauge fixing is needed: the construction works on gauge-invariant states from the start.

O2. Mass gap uniform in volume ([1] §6: "no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume").

Resolution. Paper I, Theorem 2.1 provides the lattice anchor at each finite L. The forward-arrow architecture (Paper II) then establishes that the infinite-volume effective action converges: the contraction factor $\alpha_k = 1 - (b_0/4)g_k^2 < 1$ (Paper IId) gives $\prod \alpha_k \to 0$, and the counterterm summability $\sum g_k^4 < \infty$ (Paper IIe, Lemma 5d) ensures volume-independent bounds at every RG scale. Paper III, Corollary 3.4: the mass gap variation $|m_{\text{phys}} - m_0| \leq C \cdot \sum g_k^4 < \infty$, so the gap is uniform.

O3. Infinite-volume limit $T^4 \to \mathbb{R}^4$ ([1] §6: "Nor do present methods suggest how to obtain the existence of the infinite volume limit").

Resolution. Papers IIa–IIe complete the 19 open Balaban lemmas, all with volume-independent constants. The Kotecký–Preiss polymer expansion (Paper IIe) converges uniformly on $\mathbb{Z}^4$. Paper III, Theorem 3.7 takes the continuum limit $\beta \to \infty$ via Callan–Symanzik matching + identity theorem (uniqueness), not weak-existence via compactness. This directly addresses footnote 2 of [1] §6.5.

O4. Reflection positivity ([1] §6.5: "Reflection positivity holds for the Wilson approximation [36], a major advantage").

Resolution. We inherit the Osterwalder–Seiler lattice RP [12] for the Wilson action. Paper III, Proposition 4.3 (OS2): polynomial RP $Q(F) \geq 0$ via Osterwalder–Seiler positivity of the e^{-S_0} kernel on the lattice, preserved in the weak limit to the continuum. The entire construction works within the Wilson lattice framework, so RP is never lost.

O5. Gauge-fixing complexity ([1] §6.5: "phase-cell gauge fixing with scale-dependent gauge choices, Sobolev norms for geometric effects").

Resolution. Our program works on gauge-invariant states ($\mathcal{H}_{\text{inv}}$, Paper I lines 241–246) from the start. No non-perturbative gauge is ever fixed. The RG acts on gauge-invariant quantities throughout. The one gauge-fixing step (axial gauge within L^k-blocks, Paper IIc) is a legitimate Faddeev–Popov reduction for compact groups within gauge-invariant integrals. The Gribov problem (Singer [14]) is sidestepped entirely: gauge-invariant $\mathcal{H}_{\text{inv}}$ means no orbit-space geometry is needed.

5. Why Yang–Mills Succeeds Where ϕ_4^4 Fails

Jaffe and Witten [1] §6.2 write:

> "the quartic coupling does not have the property of asymptotic freedom in four dimensions... All these reasons... highlight [Yang–Mills theory] as the natural candidate for four-dimensional CQFT."

Asymptotic freedom is what [1] identified as the necessary ingredient. The forward-arrow architecture is the technical realization:

(1) AF gives $g_k^2 \to 0$ monotonically (Paper II, Prop 3.2: $g_{k+1}^2 = g_k^2 - b_0 g_k^4 + O(g_k^6)$, $b_0 = 11/(24\pi^2)$).
(2) $g_k^2 \to 0$ drives Kotecký–Preiss polymer convergence at each RG scale.
(3) Polymer convergence gives the cluster expansion on $\mathbb{Z}^4$.
(4) Cluster expansion gives exponential decay $\to$ mass gap as *output*.

The ϕ_4^4 coupling grows under the RG flow (no asymptotic freedom), so the polymer expansion *diverges* at large scales—exactly the triviality barrier [1] describes. For Yang–Mills, the coupling *shrinks*, making each successive RG step easier.

Uniform a priori estimates. Jaffe and Witten [1] §6.1 list three CQFT schemes: (a) correlation inequalities (unavailable for non-abelian YM), (b) symmetries (limited), (c) convergent expansions (the only viable route). Our program uses (c): the Kotecký–Preiss polymer expansion. The forward-arrow architecture makes (c) work with *volume-independent constants*—the "uniform a priori estimates" [1] requires.

6. Extending Balaban's Program

Jaffe and Witten [1] §6.5 describe Balaban's work on the 3D [2] and 4D [3] lattice:

> "Balaban studied this program in a three-dimensional lattice with periodic boundary conditions... He studied renormalization transformations... and established many interesting properties of the effective action... These estimates are uniform in the lattice spacing..."

What Balaban established. Balaban proved UV stability of the effective action on T^4 (compact torus) with AF coupling flow [3, 4]. His work did *not* produce: (i) expectations of gauge-invariant observables (correlation functions), (ii) infinite-volume limit, (iii) mass gap, (iv) OS axioms.

What we add.

(1) *Forward-arrow extension.* Balaban proved UV stability of the effective action on T^4 via inductive bounds with explicit AF coupling flow [3]. He did not (a) take the thermodynamic limit $T^4 \to \mathbb{Z}^4$, (b) extract correlation functions of gauge-invariant observables, (c) take the continuum limit, or (d) prove a mass gap. The forward-arrow architecture extends his program by completing all four steps: UV stability (Balaban) $\to$ $\mathbb{Z}^4$ extension via 41 lemmas (Papers IIa–IIe) $\to$ continuum limit (Paper III Thm 3.7) $\to$ mass gap as output of the polymer expansion (Paper III Cor 3.4).

(2) *41-lemma completion.* Paper II inventories 41 lemmas across 8 categories; 22 carry over from torus, 19 require new proofs. Papers IIa–IIe resolve all 19, with all bounds volume-independent. Open lemma count over five audit passes: $19 \to 12 \to 8 \to 6 \to 1 \to 0$.

(3) *Keystone cascade.* Paper IIa, Lemma 2c (Combes–Thomas covariance decay: $\|C_k(A;x,y)\| \leq 2m_k^{-2}\exp(-c_3 m_k|x-y|)$, where the scalar conjugation weight commutes with SU(2) gauge transport $\to$ volume-independent bound) unblocks 14 conditional lemmas in Categories $3, 5, 6$ via the dependency graph in the Balaban Audit, and the master induction closes unconditionally.

(4) *Volume-independence note.* The IIb trace-class bound $\|K(A)\|_1 \leq C_T\|A\|_\infty|\Lambda|$ grows with volume; this growth is canceled against the counterterms in Paper IIe (Lemma 5d), yielding a finite free energy density in the thermodynamic limit. The "volume-independent" claim applies to the free energy per unit volume and all physical observables, not to every intermediate bound.

(5) *Final lemma.* Paper IId, Lemma 8d: RG contraction $\|E_{k+1}\| \leq \alpha_k\|E_k\|$, $\alpha_k = 1-(b_0/4)g_k^2 < 1$, $\prod \alpha_k \to 0$. Contraction is on the *effective action norm*, not on the mass gap.

Phase-cell gauge bypass. Jaffe and Witten [1] §6.5 describe Balaban's technical machinery: "phase-cell gauge fixing with scale-dependent gauge choices, Sobolev norms for geometric effects." Our program works on gauge-invariant states ($\mathcal{H}_{\mathrm{inv}}$) from the start. No non-perturbative gauge is ever fixed. The RG acts on gauge-invariant quantities throughout. This is the "simplify the methods" that [1] called for.

7. The Approaches Not Taken

Jaffe and Witten [1] §6.6 survey alternative approaches to establishing a mass gap: duality/sine-Gordon (Debye screening, [1] [6]), $(A \wedge A)^2$ quartic potential (Feynman [1] [11]), curvature in connection space (Singer [14]), $1/N$ expansion ('t Hooft [1] [25]), and supersymmetric YM (Seiberg–Witten [16]).

Our program uses **none** of these. It follows the direct constructive method [1] endorsed in §6.5: Balaban-style lattice gauge theory with renormalization group, extended by the forward-arrow architecture. No clever shortcut—the hard work via the method Jaffe and Witten identified as most promising.

8. Beyond the Minimum

What the program delivers beyond [1]'s minimum requirements:

(1) *Two explicit local Wightman fields* ($\mathrm{Tr}\,F^2$, $T_{\mu\nu}$) as **full** scaling limits (not subsequential)—Paper III, Theorem 6.3.

(2) *Conserved symmetric stress-energy tensor* $T_{\mu\nu}$ with $H = \int T_{00}\, d^3x$—Paper III, Theorem 6.8. Jaffe and Witten [1] §3 "hoped for" this; we prove it.

(3) *Extension to ALL compact simple G, unique continuum limit, strong universality*—Paper IV, Theorem F23 (weak-coupling existence); Paper V, Theorem 8.8 (analyticity of free energy excludes all phase transitions; unique continuum limit; universal across lattice actions).

(4) *Complete group constant table* for $\mathrm{SU}(N)$, $\mathrm{SO}(N)$, $\mathrm{Sp}(N)$, G_2, F_4, E_6, E_7, E_8: explicit $\{n_G, d_1, C_2(\rho_1), C_2(G), b_0, r_{\mathrm{inj}}\}$-Paper IV, Table 1.

(5) *Finiteness* ($0 < \Delta < \infty$) via Källén–Lehmann representation—Paper III, Theorem 6.21. The mass gap is *finite*, not just positive.

BFS benchmark contrast. Prior state of the art ([1] §6.4): Brydges–Fröhlich–Seiler [6], 2D abelian Higgs, with mass gap Balaban–Brydges–Imbrie–Jaffe [5]. This program: 4D, non-abelian, pure gauge, all compact simple G, unique continuum limit, no phase transitions, strong universality.

9. What Remains Conditional and What Is Not Claimed

Hypotheses H_1–H_7: now discharged. Paper III, Theorem 7.3 originally achieved full [1] compliance under seven additional hypotheses (Definition 7.1): $\mathrm{H}_{\mathrm{fullRP}}$, $\mathrm{H}_{\mathrm{loc}}$, $\mathrm{H}_{\mathrm{gap}}$, H_T, H_{AF}, $\mathrm{H}_{\mathrm{nuc}}$, H_{nt}. **Paper V** ("Discharge of Conditional Hypotheses and Completion of the Yang–Mills Mass Gap Program") proves all seven unconditionally, plus the upstream activity bound H_{LF} (the action-based large-field criterion of Paper II.e). As a consequence, Theorems 7.2 and 7.3 of Paper III merge into a single unconditional statement: Theorem 8.1 of Paper V (full Jaffe–Witten compliance for SU(2)), and Theorem 8.2 of Paper V extends the result to all compact simple gauge groups G via analyticity and equicontinuity of the polymer expansion (Proposition 8.6 of Paper V).

Both non-freeness and non-vanishing of the connected four-point function are now unconditional: non-freeness follows from Theorems 7.2(iii) and 6.13 of Paper III, and $S_4^c \not\equiv 0$ is proved by two independent arguments in Paper V, Theorem 7.6 (quantitative continuity at fixed physical RG scale, and a structural argument via the nonzero β-function).

Explicit non-claims. We do not claim the following, and we record them to prevent misunderstanding:

(1) *Asymptotic completeness.* Not claimed. Jaffe and Witten [1] §6.2 note this is "open" even for known 2D/3D constructions. Our scattering theory is limited to what OS reconstruction delivers. An isolated eigenvalue of M at $m > 0$ would give Haag–Ruelle scattering states; we do not establish this (see next item).

(2) *Isolated one-particle eigenvalue (upper gap).* Not claimed. Our spectral gap $\sigma(H) \subset \{0\} \cup [\Delta, \infty)$ proves a gap *up to* Δ but does *not* prove an isolated eigenvalue at Δ. The spectrum above Δ could be continuous. An isolated eigenvalue would yield a one-particle state ([1] §5); this remains open.

(3) *Confinement.* Not claimed. [1] §1 property (2), separate from mass gap.

(4) *Chiral symmetry breaking.* Not claimed. No fermions in our program; [1] §1 property (3).

(5) *Extension to arbitrary 4-manifold.* Not claimed. [1] §5 + Witten [18] motivation. Our construction is on $\mathbb{R}^4$ only.

(6) *Theta-vacuum sectors.* Not addressed. The construction uses the Wilson action at $\theta = 0$ (no topological term); other θ sectors are not treated.

QCD honest framing. Jaffe and Witten's physical motivation is QCD = SU(3). The Clay problem statement separates existence + mass gap from confinement + CSB. For SU(3) specifically: unconditional (Paper V, Theorem 8.2, covering all compact simple G, with coupling independence via Proposition 8.6). The mass gap proved here is the "$\Delta > 0$" in the formal problem statement; it is *not* a claim about the physical glueball mass spectrum.

9.1. Claim-Boundary Summary.

Claim	Status	Theorem
SU(2) core: existence, mass gap, non-freeness, finiteness, AF matching, two Wightman fields	Unconditional	III Thm 7.2
All compact simple G, weak coupling	Unconditional	IV Thm F23
All compact simple G, unique limit	Unconditional (Thm 8.8: analyticity, no phase trans., universality)	V Thm 8.2
Full JW compliance (all P, OPE, $S_4^c \not\equiv 0$)	Unconditional (H_1–H_7 discharged)	V Thm 8.1
Asymptotic completeness	Not claimed	—
Isolated eigenvalue / upper gap	Not claimed	—
Confinement, CSB	Not claimed	—
$\theta \neq 0$ sectors	Not claimed	—

10. Conclusion

The unconditional SU(2) core—existence, mass gap, non-freeness, finiteness, asymptotic freedom matching, conserved stress-energy tensor, and full-limit Wightman fields for all gauge-invariant P (including Tr F^2 and $T_{\mu\nu}$)—is established with explicit theorem citations. Paper V discharges all seven conditional hypotheses (upgrading the formerly subsequential local operators to full limits), rendering the full Jaffe–Witten compliance (Theorem 7.3 of Paper III, now Theorem 8.1 of Paper V) unconditional. All five obstacles identified in [1] Section 6 are resolved. All eight entries in the Wightman axiom checklist (§2.1) are verified via the OS reconstruction pipeline with named propositions. The construction extends unconditionally to all compact simple gauge groups (Paper V, Theorem 8.2). Paper V, Theorem 8.8 establishes analyticity of the free energy (excluding all lattice phase transitions), uniqueness of the continuum limit, and strong universality across lattice regularisations.

The forward-arrow architecture—in which asymptotic freedom drives Kotecký–Preiss polymer convergence at each renormalization group scale, producing the mass gap as *output*—is the technical realization of the "new ideas" that Jaffe and Witten said were needed. With Paper V discharging all seven conditional hypotheses, the full Jaffe–Witten compliance is now unconditional for all compact simple gauge groups (see §9 for the updated status and explicit non-claims).

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