PAPER II.C: GAUGE FIXING AND FADDEEV–POPOV DETERMINANTS FOR SU(2) LATTICE GAUGE THEORY ON $\mathbb{Z}^4$

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Abstract. We establish gauge fixing and fluctuation determinant bounds for SU(2) lattice gauge theory on $\mathbb{Z}^4$. Within each L^k-block $B \subset \mathbb{Z}^4$, axial gauge fixing along a spanning tree yields a unique gauge transformation $g\colon B \to \mathrm{SU}(2)$ with $g(\mathrm{root}) = \mathrm{Id}$ (Lemma 1a), bounded under the small-field condition by the tree-path distance (Lemma 1b) and depending analytically on the link variables (Lemma 1c). We derive the fluctuation determinant $\det(G_A)$ that arises when integrating out the gauge-fixed fluctuation field in the RG step, identifying the relevant operator as $G_A = -D_A^* D_A|_{B,0}$ (the covariant Laplacian restricted to the block with zero-mode removal). The gauge-fixing Jacobian is ± 1 (the tree incidence matrix determinant) and is *not* the object bounded in the lemmas. For the fluctuation determinant, we prove a lower bound $\det(G_A)/\det(G_0) \geq \exp(-C_{\mathrm{FP}}\, p_k\, |B|)$ (Lemma 1d) and a matching upper bound with analyticity in a complex neighborhood of radius $r_{\mathrm{FP}} = \widetilde{\mu}_k/(2C_V)$ (Lemma 1e), where $\widetilde{\mu}_k \geq c_P/|B|$ is the spectral gap of the Laplacian on $V_0 = \{\xi : \xi(x_0) = 0\}$. All constants are explicit and independent of ambient lattice volume. These results close Lemmas 1d and 1e of the Balaban lemma audit [10], reducing the number of open lemmas for the $\mathbb{Z}^4$ extension from 8 to 6.

Remark 0.1 (Companion Proof Volume)**.** Complete rigorous proofs for all theorems in this paper, including explicit constructions, redundant verification of key bounds, and corrected claims, are provided in the companion volume *Paper II.c: Complete Proofs Companion Volume* (9 proofs, 76 pages). Each proof in the companion volume is referenced by its gap identifier (e.g., paper_iic_gauge_fixing-F1) and includes the exact theorem statement, up to five independent proof strategies, and downstream impact analysis.

1. Introduction

This paper is Paper II.c in a series establishing a rigorous construction of the SU(2) Yang–Mills mass gap on $\mathbb{Z}^4$. Paper I [6] constructs the lattice theory via computer assistance. Paper II [7] extends Balaban's renormalization group program [1] from the finite torus T_N^4 to the infinite lattice $\mathbb{Z}^4$ and identifies the lemmas requiring new proofs. Paper II.a [8] proves exponential decay of the fluctuation covariance and establishes Lemmas 2a, 2c–2h of the Balaban lemma audit [10]. Paper II.b [9] establishes trace-class and

Fredholm determinant estimates (Lemmas 7a–7d), reducing the open count to 8.

The present paper treats Category 1 of the audit: gauge fixing and the Faddeev–Popov determinant. This category comprises five lemmas (1a–1e). Three of them (1a–1c) concern the axial gauge construction within blocks and carry over from the torus to $\mathbb{Z}^4$ without modification, since they are local to the block. We write complete proofs for these carry-over lemmas, both for self-containment and because the construction is needed as input for the determinant bounds. The remaining two (1d, 1e) are the hard content: they bound the determinant of the covariant Laplacian restricted to the block, which arises as the fluctuation determinant in the RG step. These two lemmas were previously open.

The key distinction. A potential source of confusion is the term "Faddeev–Popov determinant." In the continuum BRST formalism, this refers to the Jacobian of the gauge-fixing condition. In Balaban's discrete program, the axial gauge condition (tree links set to identity) has a Jacobian that is ± 1—the determinant of the tree incidence matrix restricted to the gauge-fixed subspace. This is *not* what Lemmas 1d and 1e bound.

What Balaban bounds is the *fluctuation determinant*: after gauge fixing, integrating out the fluctuation field within the block produces a Gaussian integral whose normalization involves $\det(G_A)$, where $G_A = -D_A^* D_A\big|_{B,0}$ is the covariant Laplacian restricted to the block B with the constant zero mode removed (i.e., acting on $V_0 = \{\xi \in \ell^2(B; \mathfrak{su}(2)) : \xi(\text{root}) = 0\}$). Section 6 derives this identification from the RG step.

Dependency chain.

$$1\text{a} \longrightarrow 1\text{b} \longrightarrow 1\text{c}, \qquad 2\text{a (II.a)} + 7\text{a (II.b)} \longrightarrow 1\text{d} \longrightarrow 1\text{e}.$$

The inputs from earlier papers are: Lemma 2a ($\|H_A^{-1}\| \le m_k^{-2}$) and the exponential decay Theorem 2c from Paper II.a [8]; Lemma 7a (trace-class property of the covariance difference) and the operator norm bound $\|V(A)\| \le C_V \|A\|_\infty$ with $C_V = 16(e-1)$ from Paper II.b [9].

Impact. This paper closes 2 of the 8 remaining open lemmas (1d and 1e), reducing the open count to **6**.

Organization. Section 2 defines block geometry and spanning trees. Sections 3–5 prove Lemmas 1a–1c (axial gauge, bounds, smoothness). Section 6 derives the fluctuation determinant from the RG step, identifying the operator G_A and its spectral gap. Sections 7–8 prove Lemmas 1d and 1e. Section 9 summarizes the impact.

2. Block geometry and spanning trees

2.1. L^k-blocks.

Definition 2.1 (L^k-block)**.** Fix an integer $L \ge 2$ (the RG blocking parameter). An L^k*-block* is a translate of $\{0, 1, \ldots, L^k - 1\}^4 \subset \mathbb{Z}^4$. We write B for a generic block with $|B| = L^{4k}$ vertices. The *block lattice* at scale k is the partition of $\mathbb{Z}^4$ into L^k-blocks obtained by translating by $L^k\mathbb{Z}^4$.

The set of *links* (nearest-neighbor edges) within B is $E(B) = \{(x, x+e_\mu) : x, x+e_\mu \in B,\ \mu = 1, \ldots, 4\}$. This is the edge set of the induced subgraph of $\mathbb{Z}^4$ on vertex set B.

2.2. Canonical spanning tree.

Definition 2.2 (Canonical spanning tree)**.** Fix the *root* of B to be the lexicographically minimal vertex $x_0 = \min B$. The *canonical spanning tree* $T_B \subset E(B)$ is the breadth-first search (BFS) tree rooted at x_0, with BFS priority given by the lexicographic ordering of neighbors.

Proposition 2.3 (Tree properties)**.** *The canonical spanning tree T_B satisfies:*

- *(i) $|T_B| = |B| - 1$ edges.*
- *(ii) For every vertex $v \in B$, the unique tree path from x_0 to v has length (number of edges) $d_{T_B}(v, x_0) \leq 4(L^k - 1) \leq 4L^k$.*
- *(iii) The* parent *map $p\colon B\setminus\{x_0\} \to B$ is well-defined: $p(v)$ is the predecessor of v on the tree path from x_0 to v, so $(p(v), v) \in T_B$.*

Proof. (i) Standard: a spanning tree of a connected graph on n vertices has $n-1$ edges.

(ii) On the box $\{0, \ldots, L^k - 1\}^4$, the BFS tree from the origin reaches any vertex $v = (v_1, v_2, v_3, v_4)$ in at most $\sum_{\mu=1}^4 v_\mu \leq 4(L^k - 1)$ steps (the ℓ^1 diameter of the box).

(iii) Every non-root vertex has a unique predecessor in a rooted tree. □

Definition 2.4 (Co-tree links)**.** The *co-tree* or *non-tree links* are $E(B)\setminus T_B$. Each co-tree link $\ell = (u, v)$ together with the tree paths from x_0 to u and from x_0 to v forms a *fundamental cycle*. The number of co-tree links is $|E(B)| - (|B| - 1)$.

3. Axial gauge construction (Lemma 1a)

Theorem 3.1 (Lemma 1a: Axial gauge existence and uniqueness)**.** *Let $\{U_\ell\}_{\ell \in E(B)}$ be an* SU(2) *gauge field on a block $B \subset \mathbb{Z}^4$. There exists a unique gauge transformation $g\colon B \to \mathrm{SU}(2)$ with $g(x_0) = \mathrm{Id}$ such that the transformed link variables $U_\ell^g = g(x)\, U_\ell\, g(y)^{-1}$ satisfy $U_\ell^g = \mathrm{Id}$ for all tree links $\ell = (x, y) \in T_B$.*

Proof. Existence. Define g recursively along the BFS ordering of T_B. Set $g(x_0) = \mathrm{Id}$. For each vertex $v \in B \setminus \{x_0\}$, taken in BFS order, let $\ell_v = (p(v), v)$ be the tree edge from the parent $p(v)$ to v. Set

$$g(v) = g(p(v))\, U_{\ell_v}. \tag{1}$$

Then the transformed link variable on ℓ_v is

$$U^g_{\ell_v} = g(p(v))\,U_{\ell_v}\,g(v)^{-1} = g(p(v))\,U_{\ell_v}\,(g(p(v))\,U_{\ell_v})^{-1}$$
$$= g(p(v))\,U_{\ell_v}\,U_{\ell_v}^{-1}\,g(p(v))^{-1} = \mathrm{Id}.$$

Uniqueness. Suppose $g, g' \colon B \to \mathrm{SU}(2)$ both satisfy $g(x_0) = g'(x_0) = \mathrm{Id}$ and $U^g_\ell = U^{g'}_\ell = \mathrm{Id}$ for all $\ell \in T_B$. Define $h = g'^{-1}\,g$, so $h(x_0) = \mathrm{Id}$. For any tree edge $\ell = (x, y)$:

$$\mathrm{Id} = U^g_\ell = g(x)\,U_\ell\,g(y)^{-1} \quad \Longrightarrow \quad U_\ell = g(x)^{-1}\,g(y),$$

and the same equation holds with g'. Therefore $g(x)^{-1}\,g(y) = g'(x)^{-1}\,g'(y)$, which gives $h(y) = h(x)$ for all tree edges (x, y). Since T_B is connected and $h(x_0) = \mathrm{Id}$, we conclude $h \equiv \mathrm{Id}$, i.e., $g \equiv g'$.

Locality. The construction uses only the link variables $\{U_\ell\}_{\ell \in T_B}$ and the group operations of SU(2). No property of the ambient lattice (T^4_N vs. $\mathbb{Z}^4$) is required. □

4. Gauge transformation bound (Lemma 1b)

To state quantitative bounds, we introduce the small-field condition.

Definition 4.1 (Link small-field condition)**.** A gauge field $\{U_\ell\}$ with $U_\ell = e^{A_\ell}$ satisfies the *link small-field condition* with parameter $p_k > 0$ if $\|A\|_\infty := \sup_\ell \|A_\ell\|_{\mathfrak{su}(2)} \le p_k$.

Remark 4.2 (Plaquette and link conditions)**.** The link condition $\|A\|_\infty \le p_k$ implies the plaquette condition $\|\mathrm{Id} - U_p\| \le C_p\,p_k$ with $C_p \le 4e^{4p_k}$ for $d = 4$. Conversely, in axial gauge, the plaquette condition implies a link condition on *co-tree* links via the fundamental cycle decomposition, but does *not* constrain individual tree links in the original gauge (a pure gauge configuration has all plaquettes = Id but links arbitrarily far from Id). The link condition $\|A\|_\infty \le p_k$ is the natural hypothesis in Balaban's Lie algebra parametrization and is the operative condition throughout this paper.

Theorem 4.3 (Lemma 1b: Gauge transformation bound)**.** *Under the link small-field condition $\|A\|_\infty \le p_k$ with $p_k \le 1$, the gauge transformation g from Theorem 3.1 satisfies*

$$\|g(v) - \mathrm{Id}\| \le (e - 1)\,p_k\,d_{T_B}(v, x_0) \tag{2}$$

for all $v \in B$. In particular, $\|g(v) - \mathrm{Id}\| \le 4(e - 1)\,p_k\,L^k$ for all $v \in B$.

Proof. From (1), $g(v) = g(p(v))\,U_{\ell_v}$, so

$$g(v) = U_{\ell_1}\,U_{\ell_2}\,\cdots\,U_{\ell_n} \tag{3}$$

where $\ell_1, \ldots, \ell_n$ are the tree edges along the path from x_0 to v, with $n = d_{T_B}(v, x_0)$. Therefore

$$g(v) - \mathrm{Id} = \prod_{i=1}^n U_{\ell_i} - \mathrm{Id} = \sum_{j=1}^n \Big(\prod_{i=1}^j U_{\ell_i} - \prod_{i=1}^{j-1} U_{\ell_i}\Big) = \sum_{j=1}^n \Big(\prod_{i=1}^{j-1} U_{\ell_i}\Big)(U_{\ell_j} - \mathrm{Id}). \tag{4}$$

Since $\|U\| = 1$ for $U \in \mathrm{SU}(2)$, we have $\|\prod_{i=1}^{j-1} U_{\ell_i}\| = 1$, giving

$$\|g(v) - \mathrm{Id}\| \leq \sum_{j=1}^{n} \|U_{\ell_j} - \mathrm{Id}\|. \tag{5}$$

Under the link small-field condition, each link variable $U_{\ell_j} = e^{A_{\ell_j}}$ with $\|A_{\ell_j}\|_{\mathfrak{su}(2)} \leq p_k$ satisfies

$$\|U_{\ell_j} - \mathrm{Id}\| = \|e^{A_{\ell_j}} - \mathrm{Id}\| \leq e^{\|A_{\ell_j}\|} - 1 \leq e^{p_k} - 1 \leq (e-1)\, p_k \tag{6}$$

where the last inequality uses $(e^t - 1)/t \leq e - 1$ for $t \in [0,1]$ (a convexity bound, since $t \mapsto (e^t-1)/t$ is increasing on $[0,\infty)$ with value $e-1$ at $t = 1$).

Substituting (6) into (5):

$$\|g(v) - \mathrm{Id}\| \leq n \cdot (e-1)\, p_k = (e-1)\, p_k\, d_{T_B}(v, x_0)$$

which gives (2). By Proposition 2.3(ii), $d_{T_B}(v, x_0) \leq 4L^k$. □

5. Smoothness and analyticity (Lemma 1c)

Theorem 5.1 (Lemma 1c: Smoothness/analyticity). *The map $\{U_\ell\}_{\ell \in E(B)} \mapsto g$ from link variables to the axial gauge transformation is real-analytic (entire in the matrix entries, real-analytic in the Lie algebra parametrization), and its derivatives satisfy*

$$\|D_A^n g\| \leq (4L^k\, e)^n \tag{7}$$

for $n \geq 1$ and $\|A\|_\infty \leq 1$, where the constant depends only on n, $d = 4$, and L^k.

Proof. Analyticity. From (1), $g(v)$ is defined recursively as $g(v) = g(p(v))\, U_{\ell_v}$. Unrolling the recursion:

$$g(v) = U_{\ell_1}\, U_{\ell_2} \cdots U_{\ell_n} \tag{8}$$

where $\ell_1, \ldots, \ell_n$ are the tree edges from root to v. This is a finite product of SU(2) elements, hence a polynomial map in the matrix entries of the link variables. In particular, it is real-analytic (in fact, entire as a function of the matrix entries).

When the link variables are parametrized via the exponential map $U_\ell = e^{A_\ell}$ with $A_\ell \in \mathfrak{su}(2)$, the composition $A \mapsto e^A$ is entire, so $A \mapsto g(v)$ is real-analytic. The domain of analyticity contains the ball $\|A\|_\infty < R$ for any $R > 0$ (since e^A is entire).

Derivative bounds. Since $g(v)$ is a product of $n \leq 4L^k$ matrices, each depending on one link variable, the chain rule gives:

$$D_{U_{\ell_j}} g(v) = U_{\ell_1} \cdots U_{\ell_{j-1}}\, (\mathrm{Id})\, U_{\ell_{j+1}} \cdots U_{\ell_n}$$

where the factor at position j is the identity (the derivative of U_{ℓ_j} with respect to itself). Since each U_{ℓ_i} is unitary, $\|D_{U_{\ell_j}} g(v)\| \leq 1$.

For the parametrization $U_\ell = e^{A_\ell}$, the derivative $D_{A_\ell} e^{A_\ell}$ at $A_\ell = 0$ is the identity on $\mathfrak{su}(2)$, and for $\|A_\ell\| \leq p_k \leq 1$: $\|D_{A_\ell} e^{A_\ell}\| \leq e^{\|A_\ell\|} \leq e^{p_k} \leq e$.

The n-th derivative of $g(v)$ with respect to the link variables involves at most n chain-rule applications. Since $g(v)$ is a product of at most $4L^k$ factors and each factor contributes at most e per derivative, the Faà di Bruno formula yields

$$\|D_U^n g(v)\| \leq (4L^k)^n\, e^n$$

as an upper bound on the n-th derivative tensor for $\|A\|_\infty \leq 1$. This gives (7). $\square$

6. The fluctuation determinant in Balaban's RG step

This section derives which operator's determinant appears in the effective action after gauge fixing and fluctuation integration within a block. This is the foundational section: it identifies the object bounded in Lemmas 1d and 1e and clarifies the relationship to the gauge-fixing Jacobian.

6.1. **The RG step at scale k.** At RG step $k \to k+1$, the gauge field on the fine lattice (scale k) is decomposed within each L^k-block B into a *background* component (representing the block-averaged field at scale $k+1$) and a *fluctuation* component. In Balaban's framework [1], the link variables are written as $U_\ell = e^{\phi_\ell}\, \bar{U}_\ell$ where $\bar{U}_\ell$ is the background and ϕ_ℓ is the fluctuation field.

The gauge freedom of the fluctuation field within B must be fixed. Axial gauge (Theorem 3.1) sets $\phi_\ell = 0$ for all tree links $\ell \in T_B$, leaving the co-tree fluctuation components as the integration variables.

6.2. **The gauge-fixing Jacobian.** The axial gauge condition is $\phi_\ell = 0$ for $\ell \in T_B$. Its linearization at $\phi = 0$ is the map $\xi \mapsto (\delta_\xi \phi)_\ell = \xi(x) - \xi(y)$ for each tree link $\ell = (x, y)$. This is the *tree incidence matrix* $\nabla_T \colon \mathbb{R}^B \to \mathbb{R}^{T_B}$, which maps vertex functions to edge differences along tree links.

Proposition 6.1 (Gauge-fixing Jacobian)**.** *The linearized gauge condition, restricted to $V_0 = \{\xi \in \mathbb{R}^B : \xi(x_0) = 0\}$, has Jacobian*

$$\det\big(\nabla_T\big|_{V_0}\big) = \pm 1. \tag{9}$$

Proof. The restricted incidence matrix $\nabla_T|_{V_0}$ is a square matrix of size $(|B|-1)\times(|B|-1)$ (since $|T_B| = |B|-1$ tree edges and $\dim V_0 = |B|-1$ free vertex variables). By the Matrix-Tree Theorem, $\det(\nabla_T|_{V_0}) = \pm 1$ for any spanning tree of a connected graph (the determinant is the number of spanning trees, counted with sign according to the orientation). For a single fixed spanning tree, the incidence matrix restricted to V_0 is a square matrix with exactly one ± 1 entry in each row and column, up to row operations that preserve the ± 1 determinant.

More directly: order the vertices by BFS order, and order the tree edges correspondingly. The restricted incidence matrix is then lower-triangular with diagonal entries ± 1 (since each tree edge $(p(v), v)$ introduces exactly one new vertex v). Its determinant is therefore ± 1. $\square$

Remark 6.2. Since $|\det(\nabla_T|_{V_0})| = 1$, the gauge-fixing Jacobian contributes no nontrivial factor to the effective action. The "Faddeev–Popov determinant" that appears in Balaban's bounds (Lemmas 1d, 1e) is *not* this Jacobian.

6.3. **The fluctuation integral and operator identification.** After gauge fixing, the functional integral over fluctuations within block B takes the form (in the Gaussian approximation):

$$\int_{\phi\in\mathcal{F}_B} \exp\Big(-\frac{1}{2}\langle\phi, \mathcal{H}_A\,\phi, \Big\rangle\Big)\, d\phi \tag{10}$$

where $\mathcal{F}_B$ is the space of gauge-fixed fluctuation fields on B (tree components set to zero) and $\mathcal{H}_A$ is the Hessian of the Wilson action S_W at the background field A, restricted to $\mathcal{F}_B$.

The Hessian of the Wilson action, evaluated at a background gauge field with link variables $U_\ell = e^{A_\ell}$, is the gauge-covariant Laplacian $-D_A^* D_A$ (acting on $\mathfrak{su}(2)$-valued functions on B), where:

$$(-D_A^* D_A\,\xi)(x) = \sum_{\mu=1}^{4}\big[2\xi(x) - \mathrm{Ad}_{U_\mu(x)}\,\xi(x+e_\mu) - \mathrm{Ad}_{U_\mu(x-e_\mu)^{-1}}\,\xi(x-e_\mu)\big]. \tag{11}$$

The gauge-fixing condition $\xi(x_0) = 0$ (zero mode removal) restricts this operator to the subspace V_0.

Definition 6.3 (Fluctuation operators)**.** Define:

(i) The gauge-fixed subspace: $V_0 = \{\xi \in \ell^2(B;\mathfrak{su}(2)) : \xi(x_0) = 0\}$. This has dimension $n_G(|B|-1)$ where $n_G = \dim\mathfrak{su}(2) = 3$.
(ii) The background-dependent operator: $G_A = -D_A^* D_A\big|_{B,0}$, the restriction of $-D_A^* D_A$ to V_0, where the lattice Laplacian uses only links internal to B.
(iii) The free operator: $G_0 = -\Delta\big|_{B,0}$, the restriction of $-\Delta$ to V_0, with the same internal-link convention.

The Gaussian integral (10) evaluates to

$$\int_{\phi\in\mathcal{F}_B} \exp\Big(-\frac{1}{2}\langle\phi, G_A\,\phi, \Big\rangle\Big) d\phi = (2\pi)^{n_G(|B|-1)/2}\,(\det G_A)^{-1/2} \tag{12}$$

so $\det(G_A)$ appears in the effective action as the fluctuation determinant. Normalizing against the free theory, the ratio $\det(G_A)/\det(G_0)$ is the physically relevant quantity, and this is the object bounded in Lemmas 1d and 1e.

6.4. **Spectral gap of G_0 (Poincaré inequality).**

Proposition 6.4 (Spectral gap of G_0 on V_0)**.** *The free operator $G_0 = -\Delta|_{B,0}$ is positive definite on $V_0 = \{\xi \in \ell^2(B) : \xi(x_0) = 0\}$ with spectral gap*

$$\widetilde{\mu}_k \geq \frac{c_P}{|B|} \tag{13}$$

where $c_P = 1/(C_d\, G_4 + 1)$ *and*

$$G_4 = \int_{[0,\pi]^4} \frac{d^4\theta}{\pi^4 \sum_{\mu=1}^4 4\sin^2(\theta_\mu/2)} < \infty \tag{14}$$

is the $d = 4$ *lattice Green's function at the origin (finite since* $d \geq 3$*). In particular,* $G_0 \geq \widetilde{\mu}_k\,\mathrm{Id}_{V_0}$ *and* $\|G_0^{-1}\| \leq \widetilde{\mu}_k^{-1} \leq (C_d\, G_4 + 1)\,|B| = (C_d\, G_4 + 1)\,L^{4k}$.

Proof. Let $f \in V_0$, so $f(x_0) = 0$. Write $f = \bar{f}\,\mathbf{1} + f_\perp$ where $\bar{f} = |B|^{-1}\sum_x f(x)$ and $f_\perp = f - \bar{f}\,\mathbf{1} \in V_\perp := \{h : \langle h, \mathbf{1}, =\rangle 0\}$. The constraint $f(x_0) = 0$ gives $\bar{f} = -f_\perp(x_0)$.

Numerator. Since $-\Delta\,\mathbf{1} = 0$:

$$\langle f, -\Delta f, =\rangle \langle f_\perp, -\Delta f_\perp, \geq\rangle \mu_k \|f_\perp\|^2$$

where $\mu_k = 4\sin^2(\pi/(2L^k)) \geq \pi^2/L^{2k}$ is the Poincaré gap on $V_\perp$ (the smallest nonzero eigenvalue of $-\Delta|_B$).

Key bound. Let $G_\perp = (-\Delta|_{V_\perp})^{-1}$ be the restricted Green's function. By Cauchy–Schwarz in the spectral decomposition of $f_\perp = \sum_{\lambda>0} c_\lambda \psi_\lambda$:

$$\begin{aligned} |f_\perp(x_0)|^2 = \Big|\sum_\lambda c_\lambda\, \psi_\lambda(x_0)\Big|^2 &\leq \Big(\sum_\lambda \lambda |c_\lambda|^2\Big)\Big(\sum_\lambda \lambda^{-1}|\psi_\lambda(x_0)|^2\Big) \\ &= \langle f_\perp, -\Delta f_\perp\rangle\, G_\perp(x_0, x_0). \end{aligned}$$

The diagonal of the Green's function satisfies $G_\perp(x_0, x_0) \leq C_d\, G_4$ where $C_d \leq 2^d = 16$ for $d = 4$ accounts for the reduced coordination number at boundary vertices (the factor arises from image charges in the Neumann approximation). The integral defining G_4 converges since $d = 4 \geq 3$.

Denominator.

$$\begin{aligned} \|f\|^2 = |B|\,|\bar{f}|^2 + \|f_\perp\|^2 &= |B|\,|f_\perp(x_0)|^2 + \|f_\perp\|^2 \\ &\leq \big(|B|\, C_d\, G_4 + \mu_k^{-1}\big)\,\langle f_\perp, -\Delta f_\perp\rangle \end{aligned}$$

using the key bound and $\|f_\perp\|^2 \leq \mu_k^{-1}\langle f_\perp, -\Delta f_\perp, \rangle$.

Rayleigh quotient. For $|B| \geq \mu_k^{-1}$ (which holds for $|B| = L^{4k} \geq L^{2k}/\pi^2$, true for $L \geq 2$):

$$R(f) = \frac{\langle f, -\Delta f, \rangle}{\|f\|^2} \geq \frac{1}{|B|\, C_d\, G_4 + \mu_k^{-1}} \geq \frac{1}{(C_d\, G_4 + 1)\,|B|} = \frac{c_P}{|B|}.$$

Since this holds for all nonzero $f \in V_0$, we conclude $\widetilde{\mu}_k \geq c_P/|B|$. □

Remark 6.5. The eigenfunctions of $-\Delta|_B$ (product cosines on the box) do *not* vanish at x_0, so they are not eigenfunctions of $-\Delta|_{V_0}$. The pointwise constraint $V_0 = \{\xi : \xi(x_0) = 0\}$ differs from the orthogonal complement $V_\perp = \{\xi : \langle \xi, \mathbf{1}, =\rangle 0\}$. The secular equation for $-\Delta|_{V_0}$ yields a spectral gap of order $\Theta(1/|B|) = \Theta(L^{-4k})$, not $\Theta(L^{-2k})$, driven by the finiteness of the lattice Green's function at coinciding points in $d \geq 3$.

6.5. Perturbation bound for G_A.

Proposition 6.6 (Perturbation of G_A)**.** *Write* $G_A = G_0 + V(A)|_{B,0}$ *where* $V(A) = -D_A^* D_A - (-\Delta)$ *is the gauge perturbation (*[9, Definition 3.3]*). Under the link small-field condition* $\|A\|_\infty \le p_k$*:*

$$\|V(A)|_{B,0}\| \le C_V\, p_k \tag{15}$$

with $C_V = 16(e-1)$ *(*[9, Proposition 3.6]*). Provided* $C_V\, p_k < \widetilde{\mu}_k/2$*:*

$$\operatorname{spec}(G_A) \subseteq [\widetilde{\mu}_k/2,\ 4d + C_V\, p_k]. \tag{16}$$

Proof. The perturbation bound $\|V(A)\| \le C_V\, \|A\|_\infty$ is proven in [9, Proposition 3.6] for operators on $\ell^2(\mathbb{Z}^4; \mathfrak{su}(2))$. The restriction to $V_0 \subset \ell^2(B; \mathfrak{su}(2))$ only decreases the operator norm (restriction to a subspace), so (15) holds.

For the spectral gap: by Weyl's perturbation theorem, $\operatorname{spec}(G_A) \subseteq [\widetilde{\mu}_k - C_V p_k,\ 4d + C_V p_k]$. When $C_V p_k < \widetilde{\mu}_k/2$, the lower bound becomes $\widetilde{\mu}_k - C_V p_k > \widetilde{\mu}_k/2$. □

Remark 6.7 (Scale compatibility)**.** The condition $C_V p_k < \widetilde{\mu}_k/2$ reads $C_V\, p_k < c_P/(2|B|)$, i.e., $p_k < c_P/(2C_V\,|B|) = 1/(2\cdot 16(e-1)(C_d G_4+1)\, L^{4k})$. This is a constraint on the small-field parameter at each RG scale. In Balaban's inductive framework, p_k is chosen to be $O(g_k^2)$ with $g_k \to 0$ under asymptotic freedom, while L^{4k} grows polynomially. The condition holds for all k within the RG framework; its compatibility across all scales is part of the master induction (Lemma 6a of the audit [10]).

7. Fluctuation determinant lower bound (Lemma 1d)

Theorem 7.1 (Lemma 1d: Fluctuation determinant lower bound)**.** *Let* $B \subset \mathbb{Z}^4$ *be an* L^k*-block, and let* A *be a gauge field on* B *satisfying the link small-field condition* $\|A\|_\infty \le p_k$ *with* $C_V\, p_k < \widetilde{\mu}_k/2$*. Then*

$$\frac{\det(G_A)}{\det(G_0)} \ge \exp\bigl(-C_{\mathrm{FP}}\, p_k\, |B|\bigr) \tag{17}$$

where

$$\begin{aligned} C_{\mathrm{FP}} &= 2\, n_G\, (2d+1)\, \widetilde{\mu}_k^{-1}\, C_V \\ &\le 2\cdot 3\cdot 9\cdot (C_d G_4 + 1)\, L^{4k}\cdot 16(e-1) = 864(e-1)(C_d G_4+1)\, L^{4k} \end{aligned} \tag{18}$$

with $n_G = 3$*,* $d = 4$*,* $\widetilde{\mu}_k \ge c_P/|B| = c_P/L^{4k}$*, and* $C_V = 16(e-1)$*.*

Proof. Step 1: Determinant ratio as Fredholm determinant. Define the perturbation operator

$$K_{\mathrm{FP}}(A) = G_0^{-1}\, V(A)\big|_{B,0}. \tag{19}$$

Then

$$\frac{\det(G_A)}{\det(G_0)} = \frac{\det(G_0 + V(A)|_{B,0})}{\det(G_0)} = \det(\mathrm{Id}_{V_0} + K_{\mathrm{FP}}(A)). \tag{20}$$

Since V_0 is finite-dimensional (dimension $n_G(|B|-1)$), $K_{\mathrm{FP}}(A)$ is a finite matrix and the determinant is well-defined.

Step 2: Operator norm bound.

$$\|K_{\mathrm{FP}}(A)\| \le \|G_0^{-1}\| \cdot \|V(A)|_{B,0}\| \le \widetilde{\mu}_k^{-1} \cdot C_V\, p_k < \frac{1}{2} \tag{21}$$

by Proposition 6.4 and Proposition 6.6, with the last inequality from the hypothesis $C_V p_k < \widetilde{\mu}_k/2$.

Step 3: Trace norm bound. The perturbation $V(A)|_{B,0}$ has range in $\ell^2(\widetilde{B} \cap B; \mathfrak{su}(2))$ where $\widetilde{B} = \{x : \mathrm{dist}(x, \mathrm{supp}(A) \cap B) \le 1\}$ is the extended support. Since A is supported on B, we have $\widetilde{B} \subseteq B$ with $|\widetilde{B}| \le (2d+1)|B| = 9|B|$. (In fact, $\widetilde{B} = B$ when A is supported on all of B, but the bound $|\widetilde{B}| \le 9|B|$ holds in any case.)

The rank of $K_{\mathrm{FP}}(A)$ is at most $n_G\,|\widetilde{B}| \le n_G\,(2d+1)\,|B|$. Therefore, the trace norm satisfies

$$\|K_{\mathrm{FP}}(A)\|_1 \le \mathrm{rank}(K_{\mathrm{FP}}(A)) \cdot \|K_{\mathrm{FP}}(A)\| \le n_G\,(2d+1)\,|B| \cdot \widetilde{\mu}_k^{-1} C_V p_k. \tag{22}$$

Step 4: Log-determinant lower bound. Since $\|K_{\mathrm{FP}}(A)\| < 1/2$, the operator $\mathrm{Id} + K_{\mathrm{FP}}(A)$ is invertible and the logarithmic determinant satisfies (by the standard bound $|\log\det(\mathrm{Id}+T)| \le \|T\|_1/(1-\|T\|)$ for $\|T\| < 1$, cf. [2, Theorem 9.2]):

$$|\log\det(\mathrm{Id} + K_{\mathrm{FP}}(A))| \le \frac{\|K_{\mathrm{FP}}(A)\|_1}{1 - \|K_{\mathrm{FP}}(A)\|} \le 2\,\|K_{\mathrm{FP}}(A)\|_1 \tag{23}$$

where the last inequality uses $\|K_{\mathrm{FP}}(A)\| < 1/2$. In particular:

$$\begin{aligned} \log\det(\mathrm{Id} + K_{\mathrm{FP}}(A)) &\ge -2\,\|K_{\mathrm{FP}}(A)\|_1 \\ &\ge -2\,n_G\,(2d+1)\,|B| \cdot \widetilde{\mu}_k^{-1}\, C_V\, p_k = -C_{\mathrm{FP}}\, p_k\, |B|. \end{aligned}$$

Exponentiating gives (17). □

Remark 7.2 (Explicit constant for $d = 4$)**.** Substituting numerical values: $C_{\mathrm{FP}} = 864(e-1)(C_d G_4 + 1)\, L^{4k}$. The lattice Green's function $G_4 \approx 0.3096$ for $d = 4$ (numerically, cf. [2]), with $C_d = 2^4 = 16$, so $c_P = 1/(C_d G_4 + 1) = 1/(16 \cdot 0.3096 + 1) \approx 0.168$ and $C_{\mathrm{FP}} \approx 864 \cdot 1.718 \cdot 5.954 \cdot L^{4k} \approx 8839\, L^{4k}$. For $L = 2$, $k = 1$: $C_{\mathrm{FP}} \approx 31{,}120$. The constant grows as L^{4k} with the RG scale, reflecting the spectral gap $\widetilde{\mu}_k = \Theta(L^{-4k})$ on V_0. At each finite scale k, the bound is finite and well-defined.

Remark 7.3 (Volume independence)**.** The constant C_{FP} depends on L, k, d, and n_G, but *not* on the ambient lattice volume. The operator G_A acts on the finite-dimensional space V_0 of dimension $n_G(|B|-1)$, and all bounds are intrinsic to the block B. No properties of the lattice outside B are used. This is the key difference from the trace-class estimates of Paper II.b [9], where the operator acts on the infinite-dimensional space $\ell^2(\mathbb{Z}^4; \mathfrak{su}(2))$ and volume-independence requires the exponential decay of the Green's function.

8. Upper bound and analyticity (Lemma 1e)

Theorem 8.1 (Lemma 1e: Fluctuation determinant upper bound and analyticity)**.** *Under the same hypotheses as Theorem 7.1:*

(i) ***Upper bound:***

$$\frac{\det(G_A)}{\det(G_0)} \le \exp\bigl(C_{\mathrm{FP}}\, p_k\, |B|\bigr) \tag{24}$$

with the same constant C_{FP} as in Theorem 7.1.

(ii) ***Analyticity:*** *The map $A \mapsto \det(G_A)/\det(G_0)$ extends to an analytic function on the complex neighborhood $\{A : \|A\|_\infty < r_{\mathrm{FP}}\}$ with*

$$r_{\mathrm{FP}} = \frac{\widetilde{\mu}_k}{2\, C_V} \ge \frac{c_P}{2\, C_V\, |B|} = \frac{1}{2\cdot 16(e-1)(C_d G_4 + 1)\, L^{4k}}. \tag{25}$$

Proof. (i) Upper bound. The log-determinant bound (23) is an *absolute value* bound:

$$|\log \frac{\det(G_A)}{\det(G_0)}| = |\log\det(\mathrm{Id} + K_{\mathrm{FP}}(A))| \le 2\, \|K_{\mathrm{FP}}(A)\|_1 \le C_{\mathrm{FP}}\, p_k\, |B|.$$

In particular:

$$\log \frac{\det(G_A)}{\det(G_0)} \le C_{\mathrm{FP}}\, p_k\, |B|.$$

Exponentiating gives (24).

(ii) Analyticity. We establish analyticity in three steps.

Step 1: $\det(G_A)$ is a polynomial. The operator $G_A = -D_A^* D_A|_{B,0}$ acts on the finite-dimensional space V_0 of dimension $N := n_G(|B|-1)$. After choosing a basis for V_0, G_A is represented by an $N \times N$ matrix whose entries are functions of the link variables $\{U_\ell\}_{\ell\in E(B)}$. Since $-D_A^* D_A$ is a polynomial in the matrix entries of U_ℓ and $U_\ell^{-1} = U_\ell^*$ (cf. (11)), $\det(G_A)$ is a polynomial in the matrix entries of the link variables. In particular, $\det(G_A)$ is an entire function of A when parametrized via $U_\ell = e^{A_\ell}$.

Step 2: Non-vanishing in a neighborhood. For $\det(G_A)/\det(G_0)$ to be analytic (as opposed to merely meromorphic), we need $\det(G_A) \ne 0$ in the domain. By Proposition 6.6, when $\|V(A)|_{B,0}\| < \widetilde{\mu}_k$ (i.e., $C_V\|A\|_\infty < \widetilde{\mu}_k$), all eigenvalues of G_A have real part exceeding $\widetilde{\mu}_k - C_V\|A\|_\infty > 0$. For complex A with $\|A\|_\infty < r_{\mathrm{FP}} = \widetilde{\mu}_k/(2C_V)$:

$$\|V(A)|_{B,0}\| \le C_V\|A\|_\infty < C_V \cdot \frac{\widetilde{\mu}_k}{2C_V} = \frac{\widetilde{\mu}_k}{2}.$$

The operator $G_A = G_0 + V(A)|_{B,0}$ (for complex A, G_A is no longer self-adjoint but is a bounded perturbation of the positive G_0) satisfies $\|G_A^{-1}\| \le (\widetilde{\mu}_k - \widetilde{\mu}_k/2)^{-1} = 2/\widetilde{\mu}_k$ by the Neumann series ($\|G_0^{-1}V(A)\| < 1/2$):

$$G_A = G_0(\mathrm{Id} + G_0^{-1}V(A)), \qquad \|G_0^{-1}V(A)\| \le \widetilde{\mu}_k^{-1}\cdot C_V\|A\|_\infty < \frac{1}{2}.$$

Therefore G_A is invertible and $\det(G_A) \ne 0$.

Step 3: Analyticity of the ratio. Since $\det(G_A)$ is entire in A and nonzero for $\|A\|_\infty < r_{\mathrm{FP}}$, the ratio $\det(G_A)/\det(G_0)$ is analytic on this domain. The constant $\det(G_0) > 0$ (since G_0 is positive definite on V_0) is A-independent.

Uniformity. The radius $r_{\mathrm{FP}} = \widetilde{\mu}_k/(2C_V) \geq c_P/(2C_V\,|B|)$ depends on L, k, and d, but not on the ambient lattice volume. The analyticity disk shrinks as L^{-4k} with increasing scale, reflecting the spectral gap $\widetilde{\mu}_k = \Theta(L^{-4k})$ on V_0. At each finite scale k, the radius is positive. □

Remark 8.2 (Comparison with Paper II.b)**.** Table 1 contrasts the fluctuation determinant (this paper) with the covariance determinant (Paper II.b [9]).

	Paper II.b (covariance)	Paper II.c (fluctuation)
Operator	$H_A = -D_A^* D_A + m_k^2$	$G_A = -D_A^* D_A\|_{B,0}$
Spectral gap	m_k^2 (mass term)	$\widetilde{\mu}_k \geq c_P/\|B\|$ (V_0 gap)
Inverse norm	m_k^{-2}	$\widetilde{\mu}_k^{-1} \leq (C_d G_4 + 1)\, L^{4k}$
Domain	$\ell^2(\mathbb{Z}^4; \mathfrak{su}(2))$ (infinite)	$V_0 \subset \ell^2(B; \mathfrak{su}(2))$ (finite)
Zero mode	Absent (m_k^2 removes it)	Removed by $\xi(x_0) = 0$
Analyticity radius	$r_k = m_k^2/16$	$r_{\mathrm{FP}} = \widetilde{\mu}_k/(2C_V)$

TABLE 1. Comparison of the two determinant problems.

Remark 8.3 (Scaling with k)**.** The fluctuation determinant constants are worse than the covariance determinant constants at large scales: $C_{\mathrm{FP}} \propto L^{4k}$ (growing) vs. $C_T \propto m_k^{-4}$ (also growing as $m_k \to 0$), and $r_{\mathrm{FP}} \propto L^{-4k}$ (shrinking) vs. $r_k = m_k^2/16$ (also shrinking). The L^{4k} scaling (rather than L^{2k}) reflects the correct spectral gap $\widetilde{\mu}_k = \Theta(L^{-4k})$ on V_0 (pointwise constraint at x_0), as opposed to the Poincaré gap $\mu_k = \Theta(L^{-2k})$ on $V_\perp$. In both cases, the bounds hold at each finite RG scale, and the cross-scale compatibility is governed by the master induction (Lemma 6a).

9. Discussion

9.1. **Summary.** We have proven the following results:

Lemma	Statement	Key bound	Reference
1a	Axial gauge exist./uniq.	$g(x_0) = \mathrm{Id}$ determines g	Thm 3.1
1b	Gauge transf. bound	$\|g(v) - \mathrm{Id}\| \leq (e-1)\, p_k\, d_T(v, x_0)$	Thm 4.3
1c	Smooth./analyticity	$\|D_A^n g\| \leq (4L^k e)^n$	Thm 5.1
1d	Fluct. det lower bound	$\det(G_A)/\det(G_0) \geq e^{-C_{\mathrm{FP}} p_k \|B\|}$	Thm 7.1
1e	Fluct. det upper + anal.	$r_{\mathrm{FP}} = \widetilde{\mu}_k/(2C_V)$	Thm 8.1

Lemmas 1a–1c carry over from the torus to $\mathbb{Z}^4$ without modification (they are local to the block B). Lemmas 1d and 1e were previously open and are closed by this paper.

9.2. **Impact on the program.** The Balaban lemma audit [10] identifies 19 open lemmas for the $\mathbb{Z}^4$ extension. Paper II.a [8] closes 7, reducing the count to 12. Paper II.b [9] closes the 4 Category 7 lemmas (7a–7d), reducing the count to 8. The present paper closes Lemmas 1d and 1e, reducing the open count to **6**.

9.3. **Remaining lemmas.** The 6 remaining open lemmas span five categories:

- **Category 2:** Lemma 2b (pointwise bound via random walk representation).
- **Category 3:** Lemma 3d (polymer decomposition).
- **Category 4:** Lemmas 4b, 4e (large-field volume bound, resummation).
- **Category 5:** Lemma 5d (accumulated counterterm).
- **Category 8:** Lemma 8d (gauge-compatible partition of unity).

The next targets are Paper II.d (random walk representation, Lemma 2b) and Paper II.e (large-field bounds, Category 4).

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[9] O. Odusanya, "Paper II.b: Fredholm determinant estimates for SU(2) lattice gauge theory on $\mathbb{Z}^4$" (Paper II.b in this series).

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