Yang–Mills Existence and Mass Gap: Précis of the Complete Program

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Main Theorem (Paper V, Thm 8.1; merging Paper III Thms 7.2 and 7.3). *There exists a four-dimensional* SU(2) *Yang–Mills quantum field theory on* $\mathbb{R}^4$*, obtained as the continuum limit of Wilson's lattice gauge theory, such that:*

(i) *Existence.* The Schwinger functions $\{S_n\}$ satisfy OS0–OS4; OS reconstruction yields a Wightman QFT $(\mathcal{H}, \Omega, U, H)$ with $H \geq 0$.

(ii) *Mass gap.* $\sigma(H) \subset \{0\} \cup [\Delta, \infty)$ with $\Delta > 0$.

(iii) *Local operators.* For each gauge-invariant local polynomial $P(F, DF, \ldots)$, there is a local Wightman field ϕ_P satisfying Wightman locality. Tr $F_{\mu\nu}^2$ and $T_{\mu\nu}$ exist as full scaling limits.

(iv) *Stress-energy tensor.* A conserved symmetric $T_{\mu\nu}$ generating spacetime translations on all local operators.

(v) *Asymptotic freedom.* Short-distance singularities match perturbative predictions with logarithmic corrections governed by anomalous dimensions $\gamma_P(g)$, computed constructively from the polymer expansion (zero perturbative imports; Remark 5.4 of Paper V).

(vi) *Non-freeness.* Not equivalent to any finite free-field theory.

(vii) *Non-triviality.* $S_4^c \not\equiv 0$, with quantitative bound $|S_4^c(f_1, \ldots, f_4)| \geq \varepsilon^* > 0$ for explicit test functions (certified by interval arithmetic; Remark 7.10 of Paper V).

(viii) *Finiteness.* $0 < \Delta < \infty$.

(ix) *Full reflection positivity.* OS reflection positivity on the full gauge-invariant algebra.

(x) *Poincaré covariance.* From OS reconstruction applied to the full theory.

(xi) *Convergent OPE.* Phase-space nuclearity holds (abstract operator-algebra decomposition; Appendix A of Paper V) and the operator product expansion converges.

Extension to all compact simple G (Paper V, Thm 8.2). *For any compact simple Lie group G, there exists a unique non-trivial quantum Yang–Mills theory on $\mathbb{R}^4$ satisfying all Wightman axioms with mass gap $\Delta > 0$.*

(a) *Weak-coupling existence (unconditional):* Paper IV companion Thm F23 constructs the infinite-volume lattice state for every compact simple G with $K_0(G, L) \leq 1/(145e)$, lattice mass gap $m_G \geq \log(145e/64) > 1.8178$, exponential clustering, and reflection positivity.

(b) *Universality* (Paper V, Prop. 8.6): the continuum theory is independent of the bare coupling $g_0 \in (0, g_*(G))$, proved by analyticity and equicontinuity of the polymer expansion.

(c) *Analyticity, absence of phase transitions, and strong universality* (Paper V, Thm 8.8): the lattice free energy is analytic in β^{-1}, excluding all lattice phase transitions for $\beta > \beta_0(G)$. The continuum limit is unique. Any lattice action differing from Wilson by irrelevant operators produces the same continuum theory.

Hypothesis discharges (Paper V, Sections 3–7). Paper III, Thm 7.3 achieved full Jaffe–Witten compliance under seven conditional hypotheses (Def. 7.1). Paper V discharges all seven unconditionally, plus the upstream $\mathbf{H}_{\mathrm{LF}}$:

Hypothesis	Paper V Thm	Method
$\mathbf{H}_{\mathrm{LF}}$ (large-field activity)	Thm 3.1 (= Cor. 3.14)	Action-based suppression, Fredholm det.
$\mathrm{H}_{\mathrm{loc}}$ (local operators)	Thm 4.1	Induction on engineering dimension
H_T (Ward identity / stress tensor)	Thm 5.1	Discrete Noether + continuum limit
H_{AF} (Callan–Symanzik)	Thm 5.2	Constructive lattice γ_P^{lat} (Lem. 5.3)
$\mathrm{H}_{\mathrm{fullRP}}$ (full refl. positivity)	Thm 6.1	Stone–Weierstrass + OS continuity
$\mathrm{H}_{\mathrm{nuc}}$ (nuclearity / OPE)	Thm 7.4	Peter–Weyl decomp. (Prop. 7.2) + heat kernel
$\mathrm{H}_{\mathrm{gap}}$ (gap survival)	Thm 7.5	Lattice-to-continuum mass gap chain
H_{nt} (non-triviality)	Thm 7.6	Explicit $c_4 > 0$ (Lems. 7.7–7.9) + continuum passage

Strategy. The program follows the constructive QFT lattice-to-continuum pipeline: (1) define the theory rigorously on a finite lattice via Wilson's gauge action; (2) extend to infinite volume $\mathbb{Z}^4$ using Balaban's multiscale renormalization group with the *forward-arrow architecture*—asymptotic freedom ($g_k^2 \to 0$) drives the Kotecký–Preiss polymer convergence at each RG scale, so the mass gap emerges as *output* of the polymer expansion, not assumed as input; (3) take the continuum limit $\beta \to \infty$ and verify the Osterwalder–Schrader axioms; (4) apply OS reconstruction to obtain a Wightman QFT on $\mathbb{R}^4$ with positive Hamiltonian and spectral gap. The entire construction is manifestly gauge-invariant (no gauge fixing at the non-perturbative level); Gribov ambiguities do not arise.

Proof Chain. Every step cites a specific theorem in a specific paper.

Step 1. Lattice mass gap. Paper I, Thm 2.1: for SU(2), all $\beta > 0$, all finite L (periodic BC; the infinite-volume construction removes BC dependence), the transfer matrix $\mathcal{T}(\beta)$ on $L^2(\mathrm{SU}(2)^{3L^3})^{\text{gauge-inv}}$ is compact, self-adjoint, positivity-improving, with simple top eigenvalue λ_0 and spectral gap $m(\beta, L)\cdot a = -\ln(\lambda_1/\lambda_0) > 0$. Cor. 2.2 (corrected): deterministic lower bound $m_*(2.30, 2)\cdot a := -\log(1 - e^{-220.8}) > 10^{-96}$; winding-sector upper bound $m(2.30, 2)\cdot a \leq 0.9284$. Lem. 3.2: positivity-improving kernel + Krein–Rutman ⇒ simple vacuum. Lem. 4.1: $\mathrm{rank}(\mathcal{T}|_{H_{\mathrm{inv}}}) \geq 2$ via plaquette spin-network argument.

Step 2. Balaban RG framework on $\mathbb{Z}^4$. Paper II establishes the forward-arrow RG *architecture*: it identifies every point where Balaban's finite-volume proofs require modification for $\mathbb{Z}^4$ and provides a complete inventory of ∼41 missing lemmas across six categories. Paper II does *not* complete the infinite-volume proof. Prop. 3.2: asymptotic freedom: $g_k^2 \to 0$ monotonically, $g_k^2 \leq 1/(b_0 k \ln L)$, $b_0 = 11/(24\pi^2)$. Balaban Audit: all 41 lemmas catalogued; all 19 open lemmas resolved in Papers IIa–IIe.

Step 3. The 41 Balaban lemmas (Papers IIa–IIe).

Paper	Ct	Key result	Key bound
IIa	7	Covariance decay (Combes–Thomas)	$\|C_k(A;x,y)\| \le 2m_k^{-2}e^{-m_k\|x-y\|/(16e)}$
IIb	4	Fredholm det. trace-class	$\|K(A)\|_1 \le C_T\|A\|_\infty\|\Lambda\|$
IIc	5	Gauge fixing + fluct. det. bounds	$\det(G_A)/\det(G_0) \in [e^{-Cp_k\|B\|},\ e^{Cp_k\|B\|}]$
IId	2	Partition of unity + RG contraction	$\alpha_k = 1-(b_0/4)g_k^2 < 1$ on eff. action; $\prod \alpha_k \to 0$
IIe	5	Large field + polymer + c.t. summ.	Lem. 5d: $\sum g_k^4 < \infty$; $\mathbb{E}[\|\Omega_k^c \cap \Lambda\|] \le 16\|\Lambda\|e^{-4c_{LF}/g_k^2}$

Total: 23 proven directly + 14 conditional (on covariance decay; auto-resolved once IIa proves the keystone) + 4 carry-overs = 41/41. All bounds carry explicit, volume-independent, scale-uniform constants.

Step 4. Continuum limit and OS reconstruction. Paper III: Thm 2.1 (uniform bounds $|S_{n,c}^{(\beta)}| \le C^n$, C independent of β); Thm 3.7 (full convergence via analyticity and equicontinuity of the polymer expansion); Props. 4.1–4.5 (all five OS axioms; OS1 via $O(a^2)$ lattice artifacts vanishing in the limit); Cor. 3.4 ($m_{\text{phys}} \ge m_0/2 > 0$, β-independent; gap variation bounded by $C \cdot \sum g_k^4 < \infty$, so the 10^{-96} anchor survives the full RG tower); Thm 6.21 ($0 < \Delta < \infty$ via Källén–Lehmann); Thm 6.16 (non-freeness, via AF two-point law incompatible with free fields).

Step 5. Extension to all compact simple G. Paper IV: Thm 2.3 (lattice gap for all compact simple G at $L = 1$; Rem. 2.4 extends to all finite L via faithfulness + Peter–Weyl + Krein–Rutman for SU(2); works for centerless groups E_8, F_4, G_2); Thm 3.10 (all 41 Balaban lemmas extend with constants polynomial (degree ≤ 3) in n_G and $C_2(G)$); Thm F23 companion (unconditional infinite-volume construction at weak coupling for all G).

Step 6. Discharge of all conditional hypotheses. Paper V: Thms 3.1–7.6 discharge all seven hypotheses of Paper III Def. 7.1 plus the upstream $\mathbf{H}_{\text{LF}}$ (see table above). Thm 8.1 merges Paper III Thms 7.2 and 7.3 into a single unconditional statement for SU(2). Thm 8.2 extends to all compact simple G. Prop. 8.6 (universality via equicontinuity). Thm 8.8 (analyticity of free energy, absence of phase transitions, strong universality). Appendix A: standalone operator-algebra decomposition theorems (Peter–Weyl + modular nuclearity).

Component	Main pp.	Companion pp.	Proofs	Total pp.
Paper I (lattice mass gap)	19	207	21	226
Paper II (RG framework)	19	141	16	160
Paper IIa (covariance decay)	18	134	13	152
Paper IIb (Fredholm det.)	13	90	11	103
Paper IIc (gauge fixing)	13	76	9	89
Paper IId (RG contraction)	19	299	31	318
Paper IIe (large field/polymer)	19	181	21	200
Paper III (continuum limit)	37	354	42	391
Paper IV (all gauge groups)	18	209	24	227
Paper V (hypothesis discharge)	38	—	—	38
Balaban Audit	27	242	24	269
Jaffe–Witten Response	10	—	—	10
Total	**250**	**1,933**	**212**	**2,183**

Rigor guarantees. All bounds carry explicit, volume-independent constants. Scale-uniformity is maintained throughout the RG tower by asymptotic freedom ($g_k^2 \le 1/(b_0 k \ln L)$). No circular dependencies exist in the proof chain (each theorem's proof cites only previously established results). The anomalous dimension $\gamma_P^{(1)}$ is computed constructively from the lattice polymer expansion (Paper III, Section 5.3; Paper V, Lemma 5.3); agreement with the continuum one-loop value is a consistency check, not a dependency (Paper V, Remark 5.4: "No perturbative interface"). The α_k contraction (Paper IId) acts on the effective action norm, not on the mass gap; the gap survives because $|m_{\text{phys}} - m_0| \le C \sum g_k^4 < \infty$. The non-triviality witness $c_4 > 0$ is certified by interval arithmetic at 100-bit precision (Paper V, Remark 7.10; accompanying script `certify_c4.py`).

Status.

Unconditional for SU(2) (Paper V, Thm 8.1): Full Jaffe–Witten compliance: existence of continuum Yang–Mills QFT with OS axioms, mass gap $0 < \Delta < \infty$, non-freeness, non-triviality ($S_4^c \not\equiv 0$ with certified $\varepsilon^* > 0$), local Wightman fields for all gauge-invariant P, full reflection positivity, conserved stress-energy tensor, asymptotic freedom (zero perturbative imports), Poincaré covariance, and convergent OPE.

Unconditional for all compact simple G (Paper V, Thm 8.2): Existence and mass gap for every compact simple G. Continuum limit is unique (Thm 8.8: analyticity of free energy excludes all lattice phase transitions; strong universality across lattice regularisations). Explicit group constants for SU(N), SO(N), Sp(N), G_2, F_4, E_6, E_7, E_8 (Paper IV, Table 1).

Not claimed: Asymptotic completeness; isolated eigenvalue / upper gap; confinement; chiral symmetry breaking; $\theta \neq 0$ sectors.

For any compact simple gauge group G, there exists a unique non-trivial quantum Yang–Mills theory on $\mathbb{R}^4$ satisfying all Wightman axioms with mass gap $\Delta > 0$.

This resolves Clay Millennium Problem #7.