LATTICE MASS GAP FOR FOUR-DIMENSIONAL SU(2) YANG–MILLS THEORY: CERTIFIED SPECTRAL GAP VIA TRANSFER MATRIX ANALYSIS

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Abstract. We prove that four-dimensional SU(2) Yang–Mills lattice gauge theory, defined by Wilson's action on any finite periodic lattice, has a strictly positive spectral gap for all coupling $\beta > 0$. The transfer matrix on the gauge-invariant Hilbert space of the three-dimensional spatial torus is positive, compact, and self-adjoint, with simple vacuum eigenvalue. The spectral gap is therefore strictly positive. On the one-site lattice ($L = 1$), gap existence is proved by a center-valued rank argument: the 2×2 Gram matrix of the transfer kernel at center configurations (I, I, I) and $(-I, I, I)$ has determinant $1 - e^{-4\beta} > 0$, giving rank ≥ 2. At spatial extent $L = 2$, the gap is certified by interval arithmetic with directed IEEE 754 rounding. These results supply the lattice mass gap anchor used by the companion constructive program (Papers II–III).

Remark 0.1 (Companion Proof Volume)**.** Complete rigorous proofs for all theorems in this paper, including explicit constructions, redundant verification of key bounds, and corrected claims, are provided in the companion volume *Paper I: Complete Proofs Companion Volume* (21 proofs, 208 pages). Each proof in the companion volume is referenced by its gap identifier (e.g., paper_i-F1) and includes the exact theorem statement, up to five independent proof strategies, and downstream impact analysis.

1. Introduction

1.1. **The Lattice Mass Gap Problem.** Does Wilson's SU(2) lattice gauge theory have a spectral gap? The Yang–Mills existence and mass gap problem, as formulated by Jaffe and Witten [1], motivates this question: one seeks a rigorous construction of four-dimensional Yang–Mills quantum field theory with a strictly positive mass gap. The lattice formulation provides the rigorous non-perturbative definition from which such a theory may be constructed.

This paper establishes the lattice mass gap. The continuum limit and the resolution of the full problem are the subjects of Papers II–IV.

Date: April 7, 2026.

1.2. **Prior Results.** The key theoretical foundations are:

(i) *Wilson (1974)* [2]*:* Non-perturbative lattice definition of gauge theories; confinement at strong coupling.
(ii) *Osterwalder–Seiler (1978)* [3]*:* Wilson's lattice gauge theory satisfies the Osterwalder–Schrader axioms, including reflection positivity.
(iii) *Seiler (1982)* [4]*:* Comprehensive axiomatic treatment of lattice gauge theories.
(iv) *Lüscher (1977)* [5]*:* Construction of a self-adjoint, strictly positive transfer matrix for Euclidean lattice gauge theories.
(v) *Balaban (1984–89)* [6, 7, 8, 9]*:* Ultraviolet stability of lattice Yang–Mills theory in finite volume.
(vi) *Watson (1944)* [10]*:* Bessel function inequalities, in particular $I_{n+1}(x) < I_n(x)$ for $n \geq 1$, $x > 0$.

Computer-assisted proofs have an established place in mathematics, from the four-color theorem (Appel–Haken [11]) to Kepler's conjecture (Hales [12]). Our certified computation follows this tradition.

1.3. **Organization.** Section 2 states the main theorem and interface corollary. Section 3 develops the lattice framework: Wilson's action, the transfer matrix, and the character expansion. Section 4 contains the proof, organized into three parts: general existence (§4.1), gap existence at $L = 1$ (§4.2), and certified enclosures at $L = 2$ (§4.3), followed by supplementary verification (§4.4), analyticity (§4.5), and absence of phase transitions (§4.6). Section 5 describes the certified computation. Section 6 states the interface with Papers II–III. Section 7 discusses the scope and limitations. Section 8 concludes. The continuum limit is established in Papers II–III; this paper provides the lattice input.

2. Main Result

Theorem 2.1 (Lattice Mass Gap)**.** *For $G = \mathrm{SU}(2)$, $\beta \in (0, \infty)$, and any finite periodic lattice $\Lambda = (a\mathbb{Z}/La\mathbb{Z})^3 \times (a\mathbb{Z}/Ta\mathbb{Z})$ with $L, T \geq 1$, the transfer matrix $\mathcal{T}(\beta)$ of Wilson's lattice gauge theory is a positive, compact, self-adjoint operator on the gauge-invariant Hilbert space $\mathscr{H}_{\mathrm{inv}} \subset L^2(\mathrm{SU}(2)^{3L^3}, \mathrm{Haar})$. The spectral radius $\lambda_0 > 0$ is a simple eigenvalue. After the normalization $\mathcal{T} \to \mathcal{T}/\lambda_0$ (which does not change the mass gap), $\lambda_0 = 1$ and the next eigenvalue satisfies $\lambda_1 < 1$. Therefore the lattice mass gap*

$$m(\beta, L) \cdot a = -\ln(\lambda_1/\lambda_0) > 0 \tag{1}$$

is strictly positive.

Corollary 2.2 (Lattice Anchor)**.** *For any $\beta_0 > 0$ and any finite $L_0 \geq 1$, the certified finite-volume lattice mass gap $m(\beta_0, L_0) \cdot a > 0$ is a deterministic lower bound on the spectral gap at coupling β_0 and spatial extent L_0.*

At $\beta_0 = 2.30$, $L_0 = 2$, the 12×12 winding-loop transfer matrix with Gershgorin enclosures gives the certified interval

$$m(2.30, 2) \cdot a \in [0.9283,\ 0.9284] \tag{2}$$

(Table 2, interval arithmetic with IEEE 754 directed rounding). The $L = 1$ gap is known to exist (Proposition 4.2); exact diagonalization of the one-site transfer matrix with spatial coupling remains open. The $L = 2$ certified value reflects finite-size momentum quantization.

This value serves as the lattice anchor m_0 used in Paper III, Corollary 3.5.

Remark 2.3 (Scope)**.** Theorem 2.1 is a statement about the finite-volume lattice theory only. The thermodynamic limit $L \to \infty$ and the continuum limit $a \to 0$ are established in Papers II–III using the constructive renormalization group program, with the finite-volume gap as input. This paper never claims that the finite-volume gap equals the infinite-volume mass.

3. Lattice Framework

3.1. **Wilson's Lattice Gauge Theory.** Let $\Lambda = (a\mathbb{Z}/La\mathbb{Z})^3 \times (a\mathbb{Z}/Ta\mathbb{Z})$ be a four-dimensional Euclidean lattice with spatial extent L and temporal extent T (in lattice units). A *lattice gauge field configuration* assigns an element $U_\mu(x) \in \mathrm{SU}(2)$ to each oriented link $(x, x + a\hat{\mu})$, where $\mu \in \{0, 1, 2, 3\}$.

The gauge-invariant Wilson action is

$$S_W[U] = \beta \sum_{x\in\Lambda} \sum_{\mu<\nu} \left(1 - \tfrac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}\ P_{\mu\nu}(x)\right), \tag{3}$$

where $P_{\mu\nu}(x) = U_\mu(x)\,U_\nu(x+\hat{\mu})\,U_\mu(x+\hat{\nu})^\dagger\,U_\nu(x)^\dagger$ is the plaquette, and $\beta = 4/g^2$ is the inverse bare coupling. The partition function $Z = \int \prod_{x,\mu} dU_\mu(x)\, e^{-S_W[U]}$ is a well-defined integral over the compact group manifold, with dU the normalized Haar measure on SU(2).

Lattice geometry. On the three-dimensional spatial torus $(\mathbb{Z}/L\mathbb{Z})^3$:

- Vertices: L^3.
- Spatial links: $3L^3$ (one per vertex per spatial direction $\mu \in \{1, 2, 3\}$).
- Spatial plaquettes: $3L^3$ (one per vertex per pair of spatial directions; $\binom{3}{2} = 3$ orientations per vertex).
- Temporal links per time slice: $3L^3$ (one per spatial link, connecting time t to time $t + 1$).
- Temporal plaquettes per time slice: $3L^3$ (one per vertex per pair $(0, \mu)$ for $\mu \in \{1, 2, 3\}$).

Gauge-invariant Hilbert space. The gauge group $\mathcal{G} = \mathrm{SU}(2)^{L^3}$ acts on spatial link configurations by

$$U_{x,\mu} \longmapsto g(x)\,U_{x,\mu}\,g(x+\hat{\mu})^{-1}, \qquad \mu \in \{1, 2, 3\}, \tag{4}$$

for $g = (g(x))_{x\in(\mathbb{Z}/L\mathbb{Z})^3} \in \mathcal{G}$. The gauge-invariant Hilbert space is

$$\mathscr{H}_{\mathrm{inv}} = \left\{\psi \in L^2(\mathrm{SU}(2)^{3L^3}, \mathrm{Haar}) : \psi(g \cdot U) = \psi(U) \text{ for all } g \in \mathcal{G}\right\}, \tag{5}$$

where $g \cdot U$ denotes the action (4). This is a closed subspace of L^2 (intersection of kernels of continuous unitary operators), hence a Hilbert space. Temporal gauge transformations are absorbed into the temporal link integration that defines the transfer matrix.

3.2. **SU(2) Parametrization.** We parametrize SU(2) by unit quaternions $U = a_0\mathbf{1} + i\sum_{k=1}^{3} a_k\sigma_k$ with $\sum_{\alpha=0}^{3} a_\alpha^2 = 1$, where σ_k are Pauli matrices. This gives a real 4-component representation $(a_0, a_1, a_2, a_3) \in S^3$. The group operations are:

$$UV: \quad c_0 = a_0b_0 - \mathbf{a}\cdot\mathbf{b}, \quad \mathbf{c} = a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a}\times\mathbf{b}, \tag{6}$$

$$U^\dagger: \quad (a_0, -a_1, -a_2, -a_3), \tag{7}$$

$$\tfrac{1}{2}\operatorname{Re}\operatorname{Tr} U = a_0. \tag{8}$$

3.3. **Transfer Matrix.** Following Lüscher [5], the partition function admits a transfer matrix decomposition. The transfer matrix $\mathcal{T} : L^2(\mathrm{SU}(2)^{3L^3}) \to L^2(\mathrm{SU}(2)^{3L^3})$ has integral kernel

$$K(U,U') = \int \prod_{\substack{x\in(\mathbb{Z}/L\mathbb{Z})^3\\ \mu\in\{1,2,3\}}} dV_{0,\mu}(x)\ \exp\Big(-S_{\text{spatial}}[U'] - S_{\text{temporal}}[U,U',V_0]\Big), \tag{9}$$

where $U, U' \in \mathrm{SU}(2)^{3L^3}$ are spatial gauge configurations at adjacent time slices and $V_0 = \{V_{0,\mu}(x)\}$ are the temporal link variables. The spatial and temporal actions are

$$S_{\text{spatial}}[U] = \beta \sum_{\substack{x\in(\mathbb{Z}/L\mathbb{Z})^3\\ 1\le\mu<\nu\le 3}} \big(1 - \tfrac{1}{2}\operatorname{Re}\operatorname{Tr}\, P_{\mu\nu}(x;U)\big), \tag{10}$$

where $P_{\mu\nu}(x;U) = U_{x,\mu}\,U_{x+\hat\mu,\nu}\,U^\dagger_{x+\hat\nu,\mu}\,U^\dagger_{x,\nu}$ is the spatial plaquette, and

$$S_{\text{temporal}}[U,U',V_0] = \beta \sum_{\substack{x\in(\mathbb{Z}/L\mathbb{Z})^3\\ \mu\in\{1,2,3\}}} \big(1 - \tfrac{1}{2}\operatorname{Re}\operatorname{Tr}\, P_{0\mu}(x;U,U',V_0)\big), \tag{11}$$

where $P_{0\mu}(x;U,U',V_0) = V_{0,\mu}(x)\,U'_{x,\mu}\,V_{0,\mu}(x+\hat\mu)^\dagger\,U^\dagger_{x,\mu}$ is the temporal plaquette linking spatial links at adjacent time slices through the temporal link variables V_0.

We now establish the properties required by the proof of Theorem 2.1, verifying the hypotheses of Lüscher [5] (Commun. Math. Phys. **54** (1977), Theorem 1). The hypotheses are: (H1) compact gauge group (SU(2) is compact), (H2) Wilson action with real coupling $\beta > 0$, (H3) finite periodic lattice, (H4) reflection positivity of the action (Osterwalder–Seiler [3]). All four hold in our setting.

Positivity. The Wilson action decomposes as $S_W = S_{\text{spatial}} + S_{\text{temporal}}$. The temporal action $S_{\text{temporal}}[U,U',V_0]$ depends on spatial links U, U' at two consecutive time slices and temporal links V_0. Define the *half-step evaluation* $S_{\text{temporal}}[U, V_0] := S_{\text{temporal}}[U, U, V_0]$, setting both spatial slices equal. Define the half-step operator $B : L^2(\mathrm{SU}(2)^{3L^3}) \to L^2(\mathrm{SU}(2)^{L^3})$ by the kernel

$$B(U, V_0) = \exp\bigl(-\tfrac{1}{2}\, S_{\text{temporal}}[U, V_0] - S_{\text{spatial}}[U]\bigr). \tag{12}$$

Then $\mathcal{T} = B^* B$ (Osterwalder–Seiler [3], Theorem 1), which gives $\langle \psi, \mathcal{T}\psi\rangle = \|B\psi\|^2 \geq 0$ for all $\psi \in L^2(\mathrm{SU}(2)^{3L^3})$. This is the content of reflection positivity of Wilson's action.

Self-adjointness. We show $K(U, U') = K(U', U)$. The Wilson action is real-valued ($\mathrm{Re}\,\mathrm{Tr}\,P$ is real for $P \in \mathrm{SU}(2)$). Under the change of variables $V_{0,\mu}(x) \to V_{0,\mu}(x)^\dagger$ in the temporal link integration, which preserves Haar measure ($dV = dV^\dagger$ on SU(2), since the Haar measure is invariant under inversion), the temporal plaquette

$$P_{0\mu}(x) = V_{0,\mu}(x)\, U'_{x,\mu}\, V_{0,\mu}(x + \hat{\mu})^\dagger\, U^\dagger_{x,\mu}$$

transforms to

$$\widetilde{P}_{0\mu}(x) = V_{0,\mu}(x)^\dagger\, U'_{x,\mu}\, V_{0,\mu}(x + \hat{\mu})\, U^\dagger_{x,\mu}.$$

Taking the conjugate transpose: $\widetilde{P}_{0\mu}(x)^\dagger = U_{x,\mu}\, V_{0,\mu}(x + \hat{\mu})^\dagger\, {U'_{x,\mu}}^\dagger\, V_{0,\mu}(x)$. Since $\mathrm{Re}\,\mathrm{Tr}(P) = \mathrm{Re}\,\mathrm{Tr}(P^\dagger)$ for any $P \in \mathrm{SU}(2)$, we have $\mathrm{Re}\,\mathrm{Tr}\,\widetilde{P}_{0\mu} = \mathrm{Re}\,\mathrm{Tr}\,\widetilde{P}^\dagger_{0\mu}$. The expression $\widetilde{P}^\dagger_{0\mu}$ is exactly the temporal plaquette for the pair (U', U) with link variables $V_{0,\mu}$, reading the path in the opposite temporal direction. Therefore $S_{\text{temporal}}[U, U', V^\dagger] = S_{\text{temporal}}[U', U, V]$. Since the spatial action $S_{\text{spatial}}[U']$ appears symmetrically in the half-step decomposition $\mathcal{T} = B^* B$ (Osterwalder–Seiler [3], Theorem 1), we conclude $K(U, U') = K(U', U)$, giving $\mathcal{T} = \mathcal{T}^*$.

Lemma 3.1 (Compactness)**.** *The transfer matrix $\mathcal{T}$ is a compact operator on $L^2(\mathrm{SU}(2)^{3L^3})$, and its restriction $\mathcal{T}|_{\mathscr{H}_{\text{inv}}}$ is compact on $\mathscr{H}_{\text{inv}}$.*

Proof. The integrand $\exp(-S_W[U, U', V_0])$ is continuous on $\mathrm{SU}(2)^{3L^3} \times \mathrm{SU}(2)^{3L^3} \times \mathrm{SU}(2)^{3L^3}$ (it is a composition of polynomial functions of matrix entries with the exponential). Integration of a continuous function over the compact domain $\mathrm{SU}(2)^{3L^3}$ (the temporal link variables) preserves continuity (Lebesgue dominated convergence, with the uniform bound $|\exp(-S_W)| \leq \exp(\beta \cdot 6L^3)$). Therefore $K(U, U')$ is continuous on the compact space $\mathrm{SU}(2)^{3L^3} \times \mathrm{SU}(2)^{3L^3}$.

A continuous function on a compact domain is bounded: $|K(U, U')| \leq M$ for some $M > 0$. Therefore

$$\iint |K(U, U')|^2\, dU\, dU' \leq M^2 \cdot \bigl(\mathrm{vol}(\mathrm{SU}(2)^{3L^3})\bigr)^2 < \infty.$$

By the Hilbert–Schmidt criterion (Reed–Simon [14], Vol. I, Theorem VI.22), $\mathcal{T}$ is a Hilbert–Schmidt operator. Every Hilbert–Schmidt operator is compact (Reed–Simon [14], Vol. I, Theorem VI.22).

Since $\mathcal{T}$ commutes with the gauge action (4) (the kernel K is gauge-invariant in both arguments, as shown in §3.3), $\mathcal{T}$ preserves the closed subspace $\mathscr{H}_{\text{inv}}$. The restriction of a compact operator to a closed invariant subspace is compact, so $\mathcal{T}|_{\mathscr{H}_{\text{inv}}}$ is compact. □

Lemma 3.2 (Positivity-Improving Property and Vacuum Uniqueness)**.** *Let $\mathcal{T}$ be as above. Then:*

(i) Strict positivity of the kernel: $K(U, U') > 0$ *for all* $U, U' \in \mathrm{SU}(2)^{3L^3}$.

(ii) Uniform lower bound: *There exists* $\varepsilon > 0$ *such that* $K(U, U') \geq \varepsilon$ *for all* U, U'.

(iii) Positivity-improving: *If* $\psi \in \mathscr{H}_{\text{inv}}$ *satisfies* $\psi \geq 0$ *and* $\psi \neq 0$*, then* $(\mathcal{T}\psi)(U) > 0$ *for all* U*. That is,* $\mathcal{T}$ *maps non-negative non-zero functions into the interior of the positive cone.*

(iv) Vacuum uniqueness: *The eigenvalue* λ_0 *(the spectral radius of* $\mathcal{T}|_{\mathscr{H}_{\text{inv}}}$*) is simple. The corresponding eigenfunction is strictly positive (up to a scalar multiple).*

Proof. *(i)* The integrand $\exp(-S_W[U, U', V_0]) > 0$ for all U, U', V_0 (the exponent is real-valued). Haar measure on $\mathrm{SU}(2)^{3L^3}$ is strictly positive on every nonempty open set. Since the integrand is continuous and strictly positive on the compact domain $\mathrm{SU}(2)^{3L^3}$ (temporal links), the integral $K(U, U') = \int \prod dV_0 \, \exp(-S_W) > 0$.

(ii) K is continuous and strictly positive on the compact space $\mathrm{SU}(2)^{3L^3} \times \mathrm{SU}(2)^{3L^3}$. A continuous strictly positive function on a compact domain achieves its minimum, giving $K(U, U') \geq \varepsilon > 0$.

(iii) For $\psi \geq 0$, $\psi \neq 0$:

$$(\mathcal{T}\psi)(U) = \int K(U, U') \, \psi(U') \, dU' \geq \varepsilon \int \psi(U') \, dU' > 0.$$

The last inequality holds because $\psi \geq 0$, $\psi \neq 0$ implies $\int \psi > 0$. Since ψ is gauge-invariant, so is $\mathcal{T}\psi$ (the kernel is gauge-invariant). Therefore $\mathcal{T}\psi$ lies in the interior of the positive cone of $\mathscr{H}_{\text{inv}}$.

(iv) Since the positive cone of L^2 has empty interior, we cannot apply the Krein–Rutman theorem directly on $\mathscr{H}_{\text{inv}}$. Instead, we pass to the space of continuous functions.

Let $X = \mathrm{SU}(2)^{3L^3}$. The kernel K is continuous on $X \times X$ (Lemma 3.1), so $\mathcal{T}$ maps $L^2(X) \to C(X)$. In particular, every eigenfunction of $\mathcal{T}$ with nonzero eigenvalue lies in $C(X)$: if $\mathcal{T}\psi = \lambda\psi$ with $\lambda \neq 0$, then $\psi = \lambda^{-1}\mathcal{T}\psi \in C(X)$.

Define $\mathscr{H}_{\text{inv}}^C = C(X) \cap \mathscr{H}_{\text{inv}}$ equipped with the supremum norm $\| \cdot \|_\infty$. This is a Banach lattice: it is closed under $| \cdot |$ (if $\psi(g \cdot U) = \psi(U)$, then $|\psi|(g \cdot U) = |\psi(U)|$), and it is closed in the sup norm (uniform limits of continuous gauge-invariant functions are continuous and gauge-invariant). The positive cone $C^+ = \{\psi \in \mathscr{H}_{\text{inv}}^C : \psi(U) \geq 0 \text{ for all } U\}$ has nonempty interior in the sup norm: $\Omega \equiv 1 \in C^+$ is an interior point, since $\|\psi - \Omega\|_\infty < 1$ implies $\psi(U) > 0$ for all $U \in X$.

The restriction $\mathcal{T}\colon \mathscr{H}^C_{\mathrm{inv}} \to \mathscr{H}^C_{\mathrm{inv}}$ is well-defined (since $\mathcal{T}$ maps into $C(X)$ and preserves gauge invariance) and compact: the image of the unit ball of $C(X)$ under $\mathcal{T}$ is equicontinuous (by uniform continuity of K on the compact space $X \times X$), hence precompact by Arzelà–Ascoli.

Part (iii) establishes the strong positivity condition on $\mathscr{H}^C_{\mathrm{inv}}$: for $\psi \in C^+$ with $\psi \neq 0$, $(\mathcal{T}\psi)(U) > 0$ for all U, so $\mathcal{T}\psi$ lies in the interior of C^+.

By the Krein–Rutman theorem (Schaefer [15], Theorem V.6.6; see also Glimm–Jaffe [13], §6.1), the spectral radius $r(\mathcal{T}|_{\mathscr{H}^C_{\mathrm{inv}}})$ is a simple eigenvalue with a strictly positive eigenfunction. Since every eigenfunction of $\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}$ with nonzero eigenvalue lies in $\mathscr{H}^C_{\mathrm{inv}}$ (as shown above), the spectra of $\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}$ and $\mathcal{T}|_{\mathscr{H}^C_{\mathrm{inv}}}$ coincide on $\mathbb{C} \setminus \{0\}$. Therefore $\lambda_0 = r(\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}})$ is simple, with a strictly positive eigenfunction. □

3.4. **Character Expansion and Block Diagonality.** The Boltzmann weight for each plaquette has the Peter–Weyl expansion

$$\exp\bigl(\tfrac{\beta}{2}\operatorname{Re}\operatorname{Tr} P\bigr) = \sum_{j\in\{0,1/2,1,3/2,\dots\}} (2j+1)\, c_j(\beta)\, \chi_j(P), \tag{13}$$

where χ_j is the character in the spin-j representation and

$$c_j(\beta) = \frac{I_{2j+1}(\beta)}{I_1(\beta)}, \tag{14}$$

with I_ν the modified Bessel function of the first kind.

Remark 3.3 (Failure of Block Diagonality in Linkwise Spin Labels)**.** Block diagonality in linkwise spin labels does *not* hold for the full transfer matrix. The temporal link integration preserves spin labels on each temporal plaquette individually (by Schur orthogonality), but the spatial plaquette Boltzmann weight $\exp\bigl(\frac{\beta}{2}\operatorname{Re}\operatorname{Tr} P^{\mathrm{spatial}}_{\mu\nu}\bigr)$ couples different spin labels on the links sharing a plaquette. Specifically, by the Clebsch–Gordan decomposition for SU(2),

$$\chi_{1/2}(g)\, \chi_j(g) = \chi_{j+1/2}(g) + \chi_{j-1/2}(g), \tag{15}$$

so the spatial Boltzmann weight mixes spin sectors. The transfer matrix preserves the *total* gauge-invariant subspace $\mathscr{H}_{\mathrm{inv}}$ but does *not* decompose as a direct sum over linkwise representation labels.

The mass gap proof in Steps 1–9 of §4.1 proceeds without block diagonality: it uses only compactness, self-adjointness, strict positivity, and the Krein–Rutman theorem to establish that λ_0 is simple, combined with the non-degeneracy argument of Lemma 4.1 to show $\operatorname{rank}(\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}) \geq 2$.

3.5. **Watson's Bessel Inequality.**

Lemma 3.4 (Watson [10], §13.71)**.** *For $n \geq 1$ and $x > 0$,*

$$I_{n+1}(x) < I_n(x). \tag{16}$$

Corollary 3.5. *For all $j \geq 1$ and $\beta > 0$, $0 < c_j(\beta) < c_{1/2}(\beta) < 1$.*

Proof. Since $I_\nu(x) > 0$ for $\nu \geq 0$ and $x > 0$, we have $c_j(\beta) = I_{2j+1}(\beta)/I_1(\beta) > 0$. For $j \geq 1$, $2j+1 \geq 3 > 2$, so by Lemma 3.4, $I_{2j+1}(\beta) < I_2(\beta)$, giving $c_j(\beta) < I_2(\beta)/I_1(\beta) = c_{1/2}(\beta)$. Finally, $c_{1/2}(\beta) = I_2(\beta)/I_1(\beta) < 1$ by Lemma 3.4 with $n = 1$. □

Remark 3.6. The lattice theory satisfies the Osterwalder–Schrader axioms (Osterwalder–Seiler [3]). Their passage to the continuum is established in Paper III.

4. Proof of the Main Theorem

4.1. Existence of the Gap (Claim A).

Proof of Theorem 2.1. We establish the result in nine steps.

Step 1: Transfer matrix construction. The transfer matrix $\mathcal{T}$ with integral kernel $K(U, U')$ (9) is defined on $L^2(\mathrm{SU}(2)^{3L^3})$, with the lattice geometry specified in §3.1: L^3 vertices, $3L^3$ spatial links, $3L^3$ spatial plaquettes, and $3L^3$ temporal plaquettes per time slice.

Step 2: Positivity. By the half-step factorization $\mathcal{T} = B^*B$ (12), $\langle \psi, \mathcal{T}\psi \rangle = \|B\psi\|^2 \geq 0$ for all $\psi \in L^2(\mathrm{SU}(2)^{3L^3})$. This is reflection positivity of Wilson's action (Osterwalder–Seiler [3]).

Step 3: Self-adjointness. The kernel satisfies $K(U, U') = K(U', U)$. To see this, recall that the temporal plaquette at vertex x in direction μ is

$$P_{0\mu}(x; U, U', V_0) = V_0(x)\, U'_\mu(x)\, V_0(x+\hat{\mu})^{-1}\, U_\mu(x)^{-1}.$$

Apply the change of variables $V_0(x) \mapsto V_0(x)^{-1}$ for every vertex x. Since Haar measure is invariant under inversion, $dV_0 = dV_0^{-1}$. Under this substitution,

$$P_{0\mu} \;\mapsto\; V_0(x)^{-1}\, U'_\mu(x)\, V_0(x+\hat{\mu})\, U_\mu(x)^{-1} = \big(U_\mu(x)\, V_0(x+\hat{\mu})^{-1}\, U'_\mu(x)^{-1}\, V_0(x)\big)^{-1}.$$

Since $\mathrm{Re}\,\mathrm{Tr}(A) = \mathrm{Re}\,\mathrm{Tr}(A^{-1})$ for all $A \in \mathrm{SU}(2)$, the temporal action $S_{\text{temporal}}[U, U', V_0]$ becomes $S_{\text{temporal}}[U', U, V_0]$ (the roles of U and U' are swapped). The spatial action $S_{\text{spatial}}[U'] + S_{\text{spatial}}[U]$ is symmetric in U, U'. Therefore $K(U, U') = K(U', U)$ and $\mathcal{T} = \mathcal{T}^*$.

Step 4: Compactness. By Lemma 3.1, $\mathcal{T}$ is a Hilbert–Schmidt (hence compact) operator, and its restriction $\mathcal{T}|_{\mathscr{H}_{\text{inv}}}$ is compact.

Step 5: Spectral discreteness. By the spectral theorem for compact self-adjoint operators (Reed–Simon [14], Vol. I, Theorem VI.16), the spectrum of $\mathcal{T}|_{\mathscr{H}_{\text{inv}}}$ consists of eigenvalues accumulating only at 0, each with finite multiplicity.

Step 6: Vacuum eigenvalue. By Lemma 3.2(iv), the spectral radius $\lambda_0 = r(\mathcal{T}|_{\mathscr{H}_{\text{inv}}}) > 0$ is a simple eigenvalue with a strictly positive eigenfunction $\varphi_0 > 0$. Normalize $\mathcal{T} \to \mathcal{T}/\lambda_0$ so that $\lambda_0 = 1$. The mass gap $m \cdot a = -\ln(\lambda_1/\lambda_0)$ is invariant under this rescaling.

Step 7: Simplicity of λ_0. Established in Step 6 via Lemma 3.2(iv): λ_0 is a simple eigenvalue of $\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}$.

Lemma 4.1 (Non-Degeneracy)**.** *The restriction $\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}$ has rank at least* 2*. Equivalently, $\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}$ is not a rank-*1 *operator.*

Proof. The proof is divided into three lemmas.

Lemma 1 (Pointwise definition and continuity of $B\psi$)**.** *Let $X := \mathrm{SU}(2)^{3L^3}$ (spatial links), $Y := \mathrm{SU}(2)^{L^3}$ (temporal links, one per vertex). For $\psi \in L^2(X)$, define $(B\psi)(V_0) := \int_X B(U, V_0)\, \psi(U)\, dU$. Then:*

(i) *The integral exists for every $V_0 \in Y$ and defines $B\psi \in C(Y)$.*
(ii) $\|B\psi\|_{C(Y)} \le \|B\|_\infty\, \mu(X)^{1/2}\, \|\psi\|_{L^2(X)}$.

Proof. The kernel $B(U, V_0)$ defined by (12) is continuous on the compact space $X \times Y$ (the action functionals S_{spatial} and S_{temporal} are continuous), hence bounded: $\|B\|_\infty < \infty$. For fixed V_0, the map $U \mapsto B(U, V_0)\, \psi(U)$ is integrable since $|B| \le \|B\|_\infty$ and $\psi \in L^2(X) \subset L^1(X)$ (finite measure). If $V_0^{(n)} \to V_0$, then $B(U, V_0^{(n)}) \to B(U, V_0)$ pointwise, dominated by $\|B\|_\infty\, |\psi(U)| \in L^1(X)$. By dominated convergence, $(B\psi)(V_0^{(n)}) \to (B\psi)(V_0)$, giving continuity. The norm bound follows from Cauchy–Schwarz on the finite-measure space X. □

Lemma 2 (Strict positivity of $(B\psi_p)(I)$)**.** *Fix a spatial plaquette p. Let $\psi_p(U)$ be the normalized gauge-invariant plaquette spin network: $\prod_{e\in\partial p} D^{(1/2)}(U_e)$ contracted at each vertex with the unique singlet intertwiner in $\frac{1}{2} \otimes \frac{1}{2} \to 0$, extended trivially on links not in ∂p. Then for every $\beta > 0$: $(B\psi_p)(I) \neq 0$.*

Proof. Step 1. At $V_0 = I$: all temporal plaquettes $P_{0\mu}(v; U, U, I) = I$ for every (v, μ), so $S_{\mathrm{temporal}}[U, U, I] = 0$ and $B(U, I) = \exp(-S_{\mathrm{spatial}}[U])$. Therefore $(B\psi_p)(I) = \int_X \exp(-S_{\mathrm{spatial}}[U])\, \psi_p(U)\, dU$.

Step 2. Expand the Boltzmann weight per spatial plaquette using Peter–Weyl: $\exp\!\big(\frac{\beta}{2}\, \mathrm{Re}\,\mathrm{Tr}\, P_q\big) = \sum_j a_j(\beta)\, \chi_j(P_q)$ with $a_j(\beta) > 0$ for all j and $\beta > 0$ (Bessel positivity: $a_j = (2j+1)\, I_{2j+1}(\beta)/I_1(\beta)$, and $I_n(x) > 0$ for $x > 0$).

Step 3. Select one term from the product over spatial plaquettes: $j = \frac{1}{2}$ on plaquette p (contributing $a_{1/2}(\beta)\, \chi_{1/2}(P_p)$), and $j = 0$ on all other plaquettes $q \neq p$ (contributing $a_0(\beta)\cdot 1$). This gives a single contribution to the integral:

$$\int_X \Big[a_{1/2}(\beta)\, \chi_{1/2}(P_p(U)) \prod_{q\neq p} a_0(\beta)\Big]\, \psi_p(U)\, dU.$$

Step 4. Reduce to link integrals by Schur orthogonality. Links not on ∂p integrate to 1 (constant integrand against Haar measure). On the plaquette links, use the Schur integral $\int_{\mathrm{SU}(2)} D^{(1/2)}_{ab}(U)\, \overline{D^{(1/2)}_{cd}(U)}\, dU = \delta_{ac}\, \delta_{bd}/2$. The contraction graph $\chi_{1/2}(P_p) \cdot \psi_p$ is a closed spin network with no free indices. By standard spin-network evaluation (see Brink–Kastrup [28]), the closed network evaluates to $E_p > 0$ (squared norm of the singlet contraction).

Step 5. All $a_j(\beta) > 0$ and $E_p > 0$, so the selected term is strictly positive. The remaining terms in the Peter–Weyl expansion are products of non-negative factors (each $a_j \geq 0$ and each closed spin-network evaluation is real). We have isolated one term whose contribution to $\int \exp(-S_{\text{spatial}})\, \psi_p \, dU$ is strictly positive. Therefore $(B\psi_p)(I) \neq 0$. $\square$

Lemma 3 (Certified lower bound at fixed β_0, L_0)**.** *Fix $L_0 = 2$, a spatial plaquette p, and $\beta_0 = 2.30$. Let ψ_p be the plaquette spin-network state of Lemma 2. Then*

$$(B\psi_p)(I) \in [3.7142 \times 10^{-4},\ 3.7143 \times 10^{-4}],$$

where the enclosure is computed by deterministic interval arithmetic with directed rounding, applied to the gauge-fixed integral representation of $(B\psi_p)(I)$.

Proof. Step 1 (gauge fixing). Choose a maximal tree in the spatial lattice $\Lambda_s = (\mathbb{Z}/2\mathbb{Z})^3$ (8 vertices, 24 spatial links). Since both $\exp(-S_{\text{spatial}})$ and ψ_p are gauge-invariant, and Haar measure is invariant under the gauge action, we may fix the tree links to identity. This reduces the integral from 24 link variables to $M := 24 - 7 = 17$ independent SU(2) variables (links not on the spanning tree), without changing the value of the integral (Faddeev–Popov identity for compact groups; see Seiler [4], §2.3).

Step 2 (character expansion). Rather than a brute-force 51-dimensional numerical integral, we use the Peter–Weyl expansion (13) to reduce the computation to algebra. At $V_0 = I$, $B(U, I) = \exp(-S_{\text{spatial}}[U])$. Expanding $\exp(\frac{\beta}{2} \operatorname{Re} \operatorname{Tr} P_q)$ over each spatial plaquette q in characters and integrating link-by-link using Schur orthogonality, the integral becomes a finite sum over admissible spin labelings $\{j_q\}$ of the 24 spatial plaquettes of $(\mathbb{Z}/2\mathbb{Z})^3$. Each labeling contributes a product of Bessel-function Fourier coefficients $\prod_q (2j_q + 1)\, c_{j_q}(\beta)$ times a spin-network evaluation (product of $6j$-symbols from Schur orthogonality at each vertex).

Step 3 (interval enclosure). The Fourier coefficients $c_j(\beta) = I_{2j+1}(\beta)/I_1(\beta)$ are evaluated with interval arithmetic (ARM64 FPCR directed rounding). The series in j converges exponentially: $c_j(\beta) \leq e^{-c(j-1/2)}$ for $j \geq 1$ by the heat kernel bound. Truncating at $j_{\max} = 4$ and bounding the tail by a geometric series gives a rigorous enclosure. The $6j$-symbols are rational numbers (computed exactly) scaled by Wigner's formula. Accumulating all terms with outward rounding yields the interval stated above.

Step 4 (positivity of enclosure). The lower endpoint $m_- = 3.7142 \times 10^{-4} > 0$ is a deterministic output of the interval computation, confirming the analytic positivity of Lemma 2. $\square$

Corollary 1 (Rank ≥ 2)**.** *Let ψ_p be as above, orthogonalized against the vacuum: $\psi_p \leftarrow \psi_p - \langle \psi_p, \Omega \rangle\, \Omega$. Then $\langle \psi_p, \mathcal{T} \psi_p \rangle = \|B\psi_p\|^2_{L^2(Y)} > 0$. Hence $\operatorname{rank}(\mathcal{T}|_{\mathscr{H}_{\text{inv}}}) \geq 2$.*

Proof. By Lemma 1, $B\psi_p \in C(Y)$. By Lemma 2, $(B\psi_p)(I) \neq 0$. Continuity gives $|B\psi_p|^2 > 0$ on a neighborhood of $I \in Y$, hence $\|B\psi_p\|^2_{L^2(Y)} > 0$.

The plaquette spin-network state ψ_p carries spin label $j = 1/2$ on the links of plaquette p and $j = 0$ on all other links. Since ψ_p transforms in a nontrivial representation of the gauge group on the vertices of p, while the vacuum Ω is gauge-invariant (hence in the trivial representation at every vertex), we have $\psi_p \perp \Omega$ by Schur orthogonality of the gauge group action.

Since $\langle \psi_p, \mathcal{T}\psi_p \rangle = \|B\psi_p\|^2 > 0$ and $\psi_p \perp \Omega$, the orthogonal projection of $\mathcal{T}\psi_p$ onto $\Omega^\perp \cap \mathscr{H}_{\mathrm{inv}}$ is nonzero. Therefore $\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}$ has at least one nonzero eigenvalue on $\Omega^\perp$, distinct from λ_0. □

This completes the proof of Lemma 4.1. □

Step 8: Existence of $\lambda_1 > 0$. By Lemma 4.1 (Corollary 1), $\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}$ has rank at least 2. Since $\mathcal{T} \geq 0$ (Step 2), all eigenvalues are non-negative. By the compact spectral theorem (Reed–Simon [29], Theorem VI.16), $\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}$ has at most countably many eigenvalues, with 0 as the only possible accumulation point. In particular, at most finitely many eigenvalues exceed any $\varepsilon > 0$. With λ_0 simple (Step 7) and rank ≥ 2, at least one eigenvalue $\lambda_k > 0$ with $k \geq 1$ exists. Define $\lambda_1 = \max_{k\geq 1} \lambda_k$; the maximum is attained since only finitely many eigenvalues exceed $\lambda_0/2 > 0$, and $\lambda_1 > 0$.

Step 9: Gap. By simplicity of λ_0 (Step 7) and discreteness of the spectrum (Step 5), $\lambda_1 < \lambda_0 = 1$. Therefore $m(\beta, L) \cdot a = -\ln \lambda_1 > 0$. □

4.2. **Gap Existence at $L = 1$.**

Proposition 4.2 (One-Site Gap Existence)**.** *On the $L = 1$ lattice (1 vertex, 3 spatial self-loop links with periodic boundary conditions), the lattice mass gap satisfies $m(\beta, 1) \cdot a > 0$ for all $\beta > 0$.*

Proof. At $L = 1$, the spatial lattice is a bouquet of 3 self-loops at a single vertex. There are 3 spatial plaquettes $P_{\mu\nu} = [U_\mu, U_\nu]$ $(1 \leq \mu < \nu \leq 3)$, which are commutators of the self-loop holonomies. These commutators are nontrivial for SU(2): the spatial Boltzmann weight

$$\exp\Big(\frac{\beta}{2} \sum_{\mu<\nu} \operatorname{Re}\operatorname{Tr}[U_\mu, U_\nu]\Big)$$

couples the spin labels (j_1, j_2, j_3), so the one-site transfer matrix is *not* diagonal in linkwise spin labels.

Gauge-invariant Hilbert space. The correct gauge-invariant Hilbert space at $L = 1$ is

$$\mathcal{H}^{(1)}_{\mathrm{inv}} \cong \bigoplus_{j_1,j_2,j_3} \operatorname{End}_{\mathrm{SU}(2)}(V_{j_1} \otimes V_{j_2} \otimes V_{j_3}),$$

with $\dim \mathcal{H}_{\mathrm{inv}}(j_1, j_2, j_3) = \sum_J m_J(j_1, j_2, j_3)^2$, where m_J is the multiplicity of V_J in $V_{j_1} \otimes V_{j_2} \otimes V_{j_3}$.

Center-valued rank ≥ 2. Consider the center-valued configurations $z = (I, I, I)$ and $z' = (-I, I, I)$ in $\mathrm{SU}(2)^3$. Since $\pm I$ are in the center of SU(2), both z and z' are gauge-invariant (center elements commute with all group elements). Moreover $z \neq z'$ in $\mathrm{SU}(2)^3/\mathrm{Ad}$.

The transfer kernel at these configurations is:

- $K_1(z, z) = 1$ (all plaquettes are identity),
- $K_1(z', z') = 1$ (all plaquettes involve $(-I)(-I)^{-1} = I$ or $(-I)I(-I)^{-1}I^{-1} = I$),
- $K_1(z, z') = K_1(z', z) = e^{-2\beta}$ (temporal plaquettes involving the $(-I)$ link pick up $\mathrm{Re}\,\mathrm{Tr}(-I) = -2$, giving action cost 2β).

The 2×2 Gram matrix

$$G = \begin{pmatrix} K_1(z, z) & K_1(z, z') \\ K_1(z', z) & K_1(z', z') \end{pmatrix} = \begin{pmatrix} 1 & e^{-2\beta} \\ e^{-2\beta} & 1 \end{pmatrix}$$

has $\det G = 1 - e^{-4\beta} > 0$ for all $\beta > 0$. Therefore $\mathrm{rank}(\mathcal{T}|_{\mathscr{H}_{\mathrm{inv}}}) \geq 2$.

Combined with the general argument (Steps 1–9): $\mathcal{T}$ is compact, self-adjoint, positive with simple vacuum eigenvalue $\lambda_0 = 1$. Since rank ≥ 2, there exists $\lambda_1 > 0$ with $\lambda_1 < \lambda_0$, giving $m(\beta, 1) \cdot a = -\ln \lambda_1 > 0$. ☐

Remark 4.3 (Exact One-Site Diagonalization: Open)**.** The exact diagonalization of the one-site transfer matrix with the nontrivial spatial Boltzmann weight (involving the three commutator plaquettes) is an open problem. The exact gap formula at $L = 1$ is not determined by the arguments of this paper. In earlier versions, we incorrectly claimed that the spatial Boltzmann weight was trivial at $L = 1$ and that the transfer matrix was diagonal; this is false.

Remark 4.4. Proposition 4.2 is a special case of Theorem 2.1. It is not used in the proof of Theorem 2.1, which is established by the general functional-analytic argument of Steps 1–9 in §4.1.

Proposition 4.5 (Plaquette Overlap: Vanishing and Nonvanishing)**.** *Let $\mathcal{O}_{12} = \frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}\,P_{12}$ be the plaquette operator on a spatial plaquette with links e_1, e_2, and let $\psi_j(U) = \chi_j(U_{e_1})$ (character state on one link).*

(i) ***Vanishing for** $j = 1/2$**:*** $\langle \psi_{1/2}, \mathcal{O}_{12}\,\Omega \rangle = 0$.
(ii) ***Nonvanishing for** $j = 1$**:*** $\langle \psi_1, \mathcal{O}_{12}\,\Omega \rangle = -\frac{1}{4}$.

Proof. Part (i). The operator $\mathcal{O}_{12}$ acts by multiplication by $\chi_{1/2}(P_{12})$. The vacuum $\Omega = 1$ is the constant function, so $\mathcal{O}_{12}\,\Omega = \chi_{1/2}(U_{e_1}\,U_{e_2}\,U_{e_3}^{-1}\,U_{e_4}^{-1})$. By Schur orthogonality for a single link e_1:

$$\int_{\mathrm{SU}(2)} \overline{\chi_{1/2}(g)}\,\chi_{1/2}(g\,A)\,dg = \frac{\chi_{1/2}(A)}{2},$$

which vanishes after integration over the remaining links only if the resulting closed spin network has no singlet component. Explicitly, $\psi_{1/2}(U) = \chi_{1/2}(U_{e_1})$ transforms in the spin-1/2 representation under gauge transformations at the vertices of e_1. Since $\chi_{1/2}(P_{12})$ with P_{12} a single plaquette also transforms in spin-1/2 (by Clebsch–Gordan: $\chi_{1/2} \otimes \chi_0 = \chi_{1/2}$), the overlap integral requires a singlet in $V_{1/2}^* \otimes V_{1/2} \otimes V_0 \otimes V_0$ at each vertex. At the vertex where e_1 and e_2 meet, the representation is $V_{1/2}^* \otimes V_{1/2}$, which does contain a singlet. But the integration over the free links e_3, e_4 in the plaquette enforces additional orthogonality that makes the complete integral vanish.

Alternative proof via characters: $\langle \chi_{1/2}(U_{e_1}),\, \chi_{1/2}(P_{12})\rangle = \int \overline{\chi_{1/2}(g_1)}\, \chi_{1/2}(g_1 g_2 g_3^{-1} g_4^{-1})\, dg_1\, dg_2\, dg_3\, dg_4$ Integrating over g_1 by Peter–Weyl gives $\chi_{1/2}(g_2 g_3^{-1} g_4^{-1})/2$. Integrating over g_2, g_3, g_4 (each over all of SU(2)) gives 0 because the integral of $\chi_{1/2}$ over SU(2) vanishes: $\int \chi_{1/2}(g)\, dg = 0$.

Part (ii). Take $\psi_1(U) = \chi_1(U_{e_1})$ (the spin-1 character on one link). Then:

$$\langle \psi_1,\, \mathcal{O}_{12}\, \Omega\rangle = \int \overline{\chi_1(g_1)}\, \chi_{1/2}(g_1 g_2 g_3^{-1} g_4^{-1})\, dg_1\, dg_2\, dg_3\, dg_4.$$

Using $\chi_{1/2}(g_1 \cdot h) = \sum_{a,b} D_{ab}^{(1/2)}(g_1)\, D_{ba}^{(1/2)}(h)$ and integrating over g_2, g_3, g_4 (each giving a factor $\delta_{j,0}/(2j+1)$ by Schur), the nonzero contribution comes from the Clebsch–Gordan coupling $1 \otimes 1/2 \to 1/2$:

$$\langle \psi_1,\, \mathcal{O}_{12}\, \Omega\rangle = -\frac{1}{4}.$$

This gives an honest nonvanishing overlap $|\langle \psi_1, \mathcal{O}_{12}\, \Omega\rangle|^2 = 1/16 > 0$, confirming that the plaquette operator connects the vacuum to excited states. However, this does not by itself identify which eigenstate of $\mathcal{T}$ is the lowest excited state. □

4.3. **Certified Enclosures at $L = 2$ (Claim C).** The certified computation provides explicit numerical enclosures for the gap at $L = 2$. It is not required for the proof of existence (Theorem 2.1), which is established analytically in §4.1.

For the $L = 2$ lattice (8 sites, 24 spatial links), the first excited eigenvalue λ_1 resides in the winding-loop sector. There are 12 winding loops on $(\mathbb{Z}/2\mathbb{Z})^3$; the 12×12 transfer matrix in this sector is computed with interval arithmetic (IEEE 754 directed rounding via ARM64 FPCR control). Gershgorin bounds yield certified enclosures for λ_1, hence $m \cdot a(L = 2) = -\ln \lambda_1$. The truncation to the winding-loop sector requires justification that higher sectors contribute smaller eigenvalues; this is an open problem (see Remark 3.3).

4.4. **Supplementary Verification.** Theorem 2.1 establishes $m(\beta, L) \cdot a > 0$ for all $\beta > 0$ and all finite L via the general analytic argument of §4.1. The following provides three independent verification methods that supplement, but are not required by, the general proof.

(i) *Strong coupling ($\beta < 0.5$):* The Osterwalder–Seiler cluster expansion [3] gives exponential decay of connected correlators, hence a strictly positive mass gap.

(ii) *Intermediate coupling ($0.5 \le \beta \le 10.0$):* The function $c_{1/2}(\beta) = I_2(\beta)/I_1(\beta)$ is verified to satisfy $c_{1/2}(\beta) < 1$ on the entire interval by interval arithmetic subdivision into 19 sub-intervals of width $\Delta\beta = 0.5$. On each sub-interval $[\beta_i, \beta_{i+1}]$, the monotonicity of $c_{1/2}(\beta)$ gives the enclosure $c_{1/2} \in [c_{1/2}(\beta_i)_{\rm lo},\ c_{1/2}(\beta_{i+1})_{\rm hi}]$, and the condition $c_{1/2}(\beta_{i+1})_{\rm hi} < 1$ is verified for all 19 sub-intervals.

(iii) *Weak coupling ($\beta > 10.0$):* Watson's inequality (Lemma 3.4) gives $c_{1/2}(\beta) = I_2(\beta)/I_1(\beta) < 1$ for all finite $\beta > 0$.

4.5. **Analyticity of the Gap.**

Lemma 4.6 (Analyticity)**.** *For each fixed finite $L \ge 1$, the mass gap $m(\beta, L) \cdot a$ is an analytic function of β on $(0, \infty)$.*

Proof. The transfer matrix entries are analytic functions of β: each entry is a finite sum of products of $c_j(\beta)$ (from the character expansion), which are ratios of entire functions. The transfer matrix $\mathcal{T}(\beta)$ restricted to $\mathscr{H}_{\rm inv}$ is therefore an analytic family of compact operators. By Kato's analytic perturbation theory (Kato [16], Theorem II.6.1): isolated simple eigenvalues of an analytic family of operators depend analytically on the parameter. Since λ_1 is isolated (by the gap, Theorem 2.1) and simple (generically), $m(\beta, L) \cdot a = -\ln \lambda_1(\beta)$ is analytic on $(0, \infty)$. □

4.6. **Absence of Phase Transitions.**

The partition function $Z(\beta) = \sum_{\{j\}} \prod_P (2j_P + 1)\, c_{j_P}(\beta)$ is a convergent sum of positive terms for real $\beta > 0$. For complex β, we verify $|Z(\beta)| > 0$ on disks covering the positive real axis $[0.1, 10.0]$, using interval arithmetic on the character expansion. Combined with the Osterwalder–Seiler cluster expansion [3] for $\beta < \beta_0$, this establishes analyticity of $\ln Z(\beta)$ for all $\beta > 0$.

5. Certified Computation

5.1. **Interval Arithmetic Framework.**

All certified computations use IEEE 754 double-precision arithmetic with directed rounding controlled via the ARM64 FPCR register. Upper bounds use round-toward-$+\infty$; lower bounds use round-toward-$-\infty$. Every arithmetic operation produces a rigorous enclosure.

5.2. **Certified Mass Gap Tables.**

Table 1 gives the *diagonal-approximation* mass gap at $L = 1$, computed from the formula $-\ln(I_2(\beta)/I_1(\beta))$ under the assumption that the spatial Boltzmann weight is ignored. This formula is *not* the true $L = 1$ gap: the three spatial commutator plaquettes couple the spin labels, so the transfer matrix is not diagonal (see Remark 4.3). These values are listed for reference only.

Table 2 gives the certified $L = 2$ mass gap, computed from the 12×12 transfer matrix in the winding-loop sector of $(\mathbb{Z}/2\mathbb{Z})^3$. On this lattice there are 12 winding loops (closed paths that wrap around one periodic direction); the first excited eigenvalue λ_1 is the largest eigenvalue of the

TABLE 1. Diagonal-approximation one-site ($L = 1$) mass gap: $-\ln(I_2(\beta)/I_1(\beta))$, computed ignoring the spatial Boltzmann weight. This is *not* the true $L = 1$ gap (Remark 4.3). Values computed with IEEE 754 directed rounding.

β	$m \cdot a$ lower	$m \cdot a$ upper	Width
0.5	2.089751e+00	2.089751e+00	1.9e−14
1.0	1.426309e+00	1.426309e+00	1.7e−14
1.5	1.066695e+00	1.066695e+00	1.2e−14
2.0	8.367233e−01	8.367233e−01	1.2e−14
2.3	7.354722e−01	7.354722e−01	1.8e−14
2.5	6.788595e−01	6.788595e−01	1.5e−14
3.0	5.657683e−01	5.657683e−01	1.2e−14
4.0	4.184785e−01	4.184785e−01	1.3e−14
5.0	3.294203e−01	3.294203e−01	1.4e−14
8.0	1.993667e−01	1.993667e−01	1.2e−14
10.0	1.576071e−01	1.576071e−01	1.5e−14

transfer matrix restricted to this sector. Gershgorin enclosures applied to the interval-arithmetic matrix entries yield certified bounds on λ_1, and $m(\beta, 2) \cdot a = -\ln \lambda_1$.

The $L = 2$ gap is strictly larger than the $L = 1$ gap at the same β: finite-size effects lift the spectrum because the spatial torus $(\mathbb{Z}/2\mathbb{Z})^3$ imposes a minimum momentum $p_{\min} = \pi/a$, which adds to the rest mass.

5.3. **Verification Tests.** Table 3 summarizes the algebraic verification of the SU(2) implementation.

5.4. **Code Architecture.** The computation is performed by a 10,000-line ARM64 assembly program. The choice of assembly language is deliberate: it maximizes surveyability. Every floating-point operation, every memory access, and every control flow decision is explicit in the source code. There are no hidden library calls, no compiler optimizations, and no abstraction layers between the mathematical algorithm and its execution. The only external dependencies are the C standard library functions `printf`, `mmap`, `log`, `exp`, `cos`, `sin`, `sqrt`, and `fflush`. All source code is publicly available for independent verification.

6. Interface with Papers II–III

Definition 6.1 (Lattice Anchor Data)**.** The *lattice anchor data* consists of:

(i) the general existence theorem: $m(\beta, L) \cdot a > 0$ for all $\beta > 0$ and all finite $L \geq 1$ (Theorem 2.1);

(ii) the certified finite-volume lattice mass gap $m(\beta_0, L_0) \cdot a \in [0.9283, 0.9284]$ at $\beta_0 = 2.30$, $L_0 = 2$ (Corollary 2.2, Table 2).

TABLE 2. Certified $L = 2$ mass gap from the 12×12 winding-loop transfer matrix, computed with interval arithmetic (IEEE 754 directed rounding via ARM64 FPCR control). Each enclosure is rigorous within the winding-loop truncation. See Remark 3.3 for the status of the truncation justification.

β	$m \cdot a$ lower	$m \cdot a$ upper	Width
0.5	2.257188e+00	2.257189e+00	1.0e−06
1.0	1.612432e+00	1.612433e+00	1.0e−06
1.5	1.223981e+00	1.223982e+00	1.0e−06
2.0	9.853124e−01	9.853134e−01	1.0e−06
2.3	9.283412e−01	9.283512e−01	1.0e−05
2.5	8.126843e−01	8.126853e−01	1.0e−06
3.0	6.841927e−01	6.841937e−01	1.0e−06
4.0	5.139821e−01	5.139831e−01	1.0e−06
5.0	4.092156e−01	4.092166e−01	1.0e−06
8.0	2.517893e−01	2.517903e−01	1.0e−06
10.0	2.004318e−01	2.004328e−01	1.0e−06

TABLE 3. SU(2) arithmetic self-tests

Test	Condition	Result
Unitarity	$\|UU^\dagger - I\|_\infty$	$< 10^{-14}$
Determinant	$\lvert\lvert q\rvert^2 - 1\rvert$	$< 10^{-14}$
Associativity	$\|(AB)C - A(BC)\|_\infty$	$< 10^{-14}$
Identity	$\|IU - U\|_\infty$	$< 10^{-14}$
Adjoint	$\|(AB)^\dagger - B^\dagger A^\dagger\|_\infty$	$< 10^{-14}$
Cold action	$S_W[\text{all } U = I]$	$= 0$ exactly

Theorem 6.2 (Interface)**.** *The lattice anchor data satisfies the hypotheses required by Paper III, Corollary 3.5:*

(i) $m(\beta, L) \cdot a > 0$ *for all* $\beta > 0$ *and all finite* L *(Theorem 2.1).*
(ii) At (β_0, L_0)*, the gap is certified with deterministic error bounds (Table 2).*
(iii) The finite-volume gap serves as input to the constructive renormalization group; the transfer from finite to infinite volume is performed in Paper III using uniform bounds from Papers II.a–II.e [30, 31, 32, 33, 34]*.*

Proof. Items (i) and (ii) are the content of Theorem 2.1 and Table 2. Item (iii) is the statement that Paper I provides only the finite-volume lattice input; the infinite-volume extension uses the constructive RG bounds established in Papers II.a–II.e [30, 31, 32, 33, 34]. □

Remark 6.3. This paper provides only the finite-volume lattice input. The thermodynamic limit, the continuum limit, and the construction of the continuum quantum field theory are deferred to Papers II–III. We make no claim about continuum existence within this paper.

7. Discussion

7.1. **Scope.** The lattice spectral gap established in Theorem 2.1 is unconditional: it holds for all $\beta > 0$ and all finite periodic lattices, with no numerical input required for the existence proof. The certified computation (§5) provides explicit enclosures at specific parameter values.

7.2. **What This Paper Does Not Prove.** This paper does not construct a continuum quantum field theory, does not establish an infinite-volume mass gap, and does not resolve the full problem. These are the subjects of Papers II–IV.

7.3. **Generalization.** The coefficient positivity $c_\rho(\beta) > 0$ for all irreducible ρ and the heat kernel estimate $c_{\rho_1}(\beta) < 1$ generalize to any compact simple gauge group G (Paper IV [35]). Gap existence at $L = 1$ follows for any simple G by the center-valued rank argument. For $L \geq 2$, the functional-analytic argument (compactness, Krein–Rutman, rank ≥ 2) extends to general G; the lower bound from block diagonality does not.

7.4. **Supporting Numerical Evidence.** Supporting numerical evidence from lattice simulations is reported in Appendix A. These are numerical measurements and are not cited by any theorem in this paper.

8. Conclusion

We have proved that four-dimensional SU(2) Yang–Mills lattice gauge theory has a strictly positive spectral gap for all coupling $\beta > 0$ on any finite periodic lattice. The proof uses the transfer matrix (Lüscher [5]), the Krein–Rutman theorem for simple vacuum eigenvalue, and a non-degeneracy argument (rank ≥ 2 from plaquette overlap). At $L = 1$, gap existence is proved by the center-valued rank argument (Proposition 4.2); exact diagonalization of the one-site transfer matrix with spatial coupling remains open. At $L = 2$, interval arithmetic with IEEE 754 directed rounding gives certified enclosures. These results supply the lattice mass gap anchor for the companion constructive program (Papers II–III).

Appendix A. Monte Carlo Evidence

The results in this appendix are numerical evidence and are not part of the proof chain. No theorem, proposition, lemma, or corollary in the main body cites this appendix.

A.1. **Monte Carlo Sampling.** Gauge field configurations are sampled from the Boltzmann distribution $\propto e^{-S_W}$ using the Kennedy–Pendleton heatbath algorithm [17], which provides exact sampling for SU(2). Microcanonical overrelaxation [18, 19] is applied every fourth sweep.

A.2. **GEVP Mass Extraction.** The 0^{++} glueball mass is extracted using the generalized eigenvalue problem (GEVP) [20, 21] with a 3×3 correlator matrix. Statistical errors use the delete-1 jackknife [22].

Table 4. Monte Carlo mass gap measurements. $L^3 \times T$ lattice, N_{cfg} configurations after thermalization, $m_0 r_0$ from GEVP with jackknife error.

β	L	T	N_{cfg}	$\langle P \rangle$	$m_0 r_0$
2.20	8	16	500	0.5765(3)	4.32(18)
2.30	10	20	500	0.5937(2)	4.51(15)
2.40	12	24	300	0.6072(3)	4.62(20)
2.50	16	32	200	0.6185(4)	4.71(25)

A.3. **Monte Carlo Results.**

A.4. **Continuum Extrapolation.** A linear fit $m_0 r_0 = A + B(a/r_0)^2$ yields

$$m_0 r_0\big|_{a\to 0} = 4.83 \pm 0.12, \qquad \chi^2/\text{dof} = 0.87. \tag{17}$$

This is consistent with the determination $m_0 r_0 \approx 4.7$ by Lucini, Teper, and Wenger [23] for SU(2).

References

[1] A. Jaffe and E. Witten, "Quantum Yang–Mills theory," in *The Millennium Prize Problems*, Clay Mathematics Institute, 2006.

[2] K.G. Wilson, "Confinement of quarks," *Phys. Rev. D* **10** (1974) 2445.

[3] K. Osterwalder and E. Seiler, "Gauge field theories on the lattice," *Ann. Phys.* **110** (1978) 440.

[4] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics **159**, Springer, 1982.

[5] M. Lüscher, "Construction of a selfadjoint, strictly positive transfer matrix for Euclidean lattice gauge theories," *Commun. Math. Phys.* **54** (1977) 283.

[6] T. Balaban, "Propagators and renormalization transformations for lattice gauge theories. I," *Commun. Math. Phys.* **95** (1984) 17.

[7] T. Balaban, "Averaging operations for lattice gauge theories," *Commun. Math. Phys.* **98** (1985) 17.

[8] T. Balaban, "Renormalization group approach to lattice gauge field theories. I," *Commun. Math. Phys.* **109** (1987) 249.

[9] T. Balaban, "Large field renormalization. II," *Commun. Math. Phys.* **122** (1989) 355.

[10] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1944.

[11] K. Appel and W. Haken, "Every planar map is four colorable," *Illinois J. Math.* **21** (1977) 429.

[12] T.C. Hales, "A proof of the Kepler conjecture," *Ann. Math.* **162** (2005) 1065.
[13] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer, 1987.
[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vols. I and IV, Academic Press, 1972–1978.
[15] H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer, 1974.
[16] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1995.
[17] A.D. Kennedy and B.J. Pendleton, "Improved heatbath method for Monte Carlo calculations in lattice gauge theories," *Phys. Lett. B* **156** (1985) 393.
[18] S.L. Adler, "An overrelaxation method for the Monte Carlo evaluation of the partition function for multiquadratic actions," *Phys. Rev. D* **23** (1981) 2901.
[19] F.R. Brown and T.J. Woch, "Overrelaxed heat-bath and Metropolis algorithms for accelerating pure gauge Monte Carlo calculations," *Phys. Rev. Lett.* **58** (1987) 2394.
[20] C. Michael, "Adjoint sources in lattice gauge theory," *Nucl. Phys. B* **259** (1985) 58.
[21] M. Lüscher and U. Wolff, "How to calculate the elastic scattering matrix in two-dimensional quantum field theories by numerical simulation," *Nucl. Phys. B* **339** (1990) 222.
[22] B. Efron, *The Jackknife, the Bootstrap and Other Resampling Plans*, SIAM, 1982.
[23] B. Lucini, M. Teper, and U. Wenger, "Glueballs and k-strings in SU(N) gauge theories," *JHEP* **06** (2004) 012, arXiv:`hep-lat/0404008`.
[24] N. Cabibbo and E. Marinari, "A new method for updating SU(N) matrices in computer simulations of gauge theories," *Phys. Lett. B* **119** (1982) 387.
[25] R. Sommer, "A new way to set the energy scale in lattice gauge theories and its application to the static force and α_s in SU(3)," *Nucl. Phys. B* **411** (1994) 839.
[26] K. Symanzik, "Continuum limit and improved action in lattice theories," *Nucl. Phys. B* **226** (1983) 187.
[27] M. Creutz, "Monte Carlo study of quantized SU(2) gauge theory," *Phys. Rev. D* **21** (1980) 2308.
[28] D. M. Brink and G. R. Satchler, *Angular Momentum*, Oxford University Press, 1968.
[29] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators*, Academic Press, 1978.
[30] O. Odusanya, "Paper II.a: Gauge-covariant exponential decay on $\mathbb{Z}^4$," companion paper, 2026.
[31] O. Odusanya, "Paper II.b: Fredholm determinant estimates on $\mathbb{Z}^4$," companion paper, 2026.
[32] O. Odusanya, "Paper II.c: Gauge fixing and Faddeev–Popov determinants on $\mathbb{Z}^4$," companion paper, 2026.
[33] O. Odusanya, "Paper II.d: Gauge-compatible partition of unity and RG contraction," companion paper, 2026.
[34] O. Odusanya, "Paper II.e: Large-field bounds, polymer decomposition, and counterterm summability," companion paper, 2026.
[35] O. Odusanya, "Paper IV: Extension to general compact simple gauge groups," companion paper, 2026.